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PARASTROPHIC SYMMETRY IN QUASIGROUP THEORY

In this article some problems in quasigroup theory are emphasized. Special attention is devoted to parastrophic symmetry. Definitions of parastrophy are defined on: relations and operations, quasigroup varieties, propositions, quasigroup identities, concepts in quasigroup theory and etc. As an example, parastrophically closed set of concepts of one-sided neutral elements as well as the corresponding varieties are given.

Key words: *quasigoroup, loop, symmetry, neutral element, parastrophe, variety, truss, bunch, identity, functional equation*

Introduction

The quasigroup theory has some different directions of investigation: *combinatorial* (Latin squares, cubes and hypercubes,. . .), *algebraic* (classes of quasigroups, homomorphisms, subquasigroups,. . .) *geometrical* (webs, configurations, . . .) and so on. Here, in this article, we pay your attention to a direction which can be called a *function direction* or a *function method*.

Every n-ary quasigroup can be defined as a pair $(Q; f)$, where Q is a set called *carrier* and f is an invertible n-ary operation defined on Q. 'Invertible' means that f is an invertible element in i-th symmetric monoid $(\Omega_n; \underset{i}{\oplus})$ for every $i = 1, \dots, n$, where $\underset{i}{\oplus}$ is called i*-th multiplication* or i*-th Mann superposition* of n-ary operations and is defined by

$$\left(f \underset{i}{\oplus} g\right)(x_1, \dots, x_n) := f(x_1, \dots, x_{i-1}, g(x_1, \dots, x_n), x_{i+1}, \dots, x_n). \tag{1}$$

Note when $n = 1$, the set of invertible operations coincides with the set of permutations of Q. But if $n > 1$, the set of invertible operations does not form a subgroup in any of the monoids $(\Omega_n; \underset{i}{\oplus})$.

Composition, that is consecutive application of multiary functions, is a binary operation on unary functions, but it is not an operation on multiary functions because arity of the composition is not defined. That is why superpositions are operations which are certain restrictions of the composition.

Now we give a very short information about directions of the investigation of invertible functions of different arities.

Problem 1. ***What is the most convenient algebra of functions for the study of invertible operations?*** Only repetition-free composition of invertible operations is invertible. From this point of view, it is convenient to study bi-unary semigroups $(\Phi; *, \xi, \tau)$, where $*, \xi, \tau$ are taken from the signature of Post algebras [9]; or position algebras [5].

Problem 2. ***Conditions under which a repetition composition of invertible operations is invertible.*** For Mann superpositions of binary operations the question was answered by V.D. Belousov [2], and for multiary functions it has been solved in [14].

For every $\sigma \in S_n$ there exists a unary superposition called σ*-parastrophism.* σ-parastrophe of an invertible operation is invertible. The author does not know any results concerning other superpositions.

Problem 3. ***Parastrophic symmetry relations in quasigroup theory.*** Every parastrophe of a quasigroup is a quasigroup, i.e., the class of all quasigroups is parastrophically closed. Hence, every concept given for all quasigroups is given for all pairwise parastrophic quasigroups. From a different point of view, a concept for a quasigroup transforms into another concept in each parastrophe of the quasigroup. These concepts are said to be *parastrophic*. It means that defining a concept in the class of all n-ary quasigroups we define $n!$ parastrophic concepts simultaneously. Since S_{n+1} defines an action on the set of all quasigroup operations with a fixed carrier and the concepts based on quasigroup operations, therefore we can define an action of S_{n+1} on arbitrary set of pairwise parastrophic concepts. These actions will help us to improve quasigroup theory. For example, to give an answer to the following questions: What dependencies exist among parastrophic concepts? How many of parastrophic concepts could be different? Analogical questions arise about propositions, varieties, and so on.

As an example, we analyse the notion of one-sided, two-sided and three-sided neutrality of an element in a quasigroup and the corresponding varieties.

Another approach to parastrophic symmetry one can find in [12].

Problem 4: ***Description of parastrophically irreducible identities (functional equations).*** An identity is said to be: *reducible*, if it is equivalent to a conjunction of identities of shorter lengths; *parastrophically reducible*, if at least one of parastrophes of the identity is reducible. For example, it is proved [15] that every quadratic quasigroup identity in $n > 4$ individual variables is reducible.

Problem 5: ***Description of relations between isotopy and isomorphy in different classes of quasigroups.*** For example, 1) if two inverse property loops are isotopic then they are pseudo-isomorphic [14]; 2) if two commutative inverse property loops are isotopic then they are isomorphic [14]; 3) a loop has isotopy-isomorphy property (all loops being isotopic to it is isomorphic) if and only if every its element is left and right companion of its pseudo-automorphism.

Problem 6: ***Let K_1 and K_2 be two classes of functions. The study of presentations of functions from K_1 as compositions of functions from K_2.*** The following sub-problems can be highlighted in this problem:

- *possibility* — when a function from K_1 can be presented as a composition of functions from K_2?
- *uniqueness* — what dependencies exist among the presentations of the same function?
- *canonicality* — is there some type of presentations which always exists and unique in some sense?
- *tools of investigation* — what dependence exists between an investigation tool of a function and the corresponding tool of its presentation components?

We cite a few examples of the results.

Possibility. 1. Belousov's theorem: *Every distributive quasigroup is linearly isotopic to a commutative Moufang loop* [14]); 2. Bruck-Toyoda theorem: *every medial quasigroup is linearly isotopic to an Abelian group* [10]; 3. Every n-ary partially associative quasigroup is a polynomial in an algebra $(Q; +, g, \varphi, a)$, where $(Q; +)$ is a group, φ its automorphism, $(Q; g)$ is a multiary *quasigroup* [1, 13]; 4. A.N. Kolmogorov [7]: *Every multiary real continuous function is presentable as a composition of unary real continuous functions and the addition.*

Uniqueness. A.V.Kuznetsov [8]: 1. *Every two full repetition-free decompositions of boolean function without dummy variable are almost the same.* In [19] the result was obtained for functions defined on three-element sets. For arbitrary power sets it is proved that the corresponding components of two full repetition-free decompositions of a quasigroup are isotopic [1] and the decompositions are almost identical [18].

Tools of investigation. Two distributive quasigroups being defined on the same Moufang loop are isomorphic if and only if their defining automorphisms are conjugated by an automorphism of the Moufang loop [14].

Problem 7: ***Description of functional equations on quasigroup operations.*** A functional equation is called *generalized* if all its functional variables are pairwise different. Therefore, some properties being true for generalized functional equations are true for all functional equations. We define equivalence relation on functional equations: two of them are equivalent if one of them can be obtained from the other in a finite number of some fixed transformations. These transformations preserve continuity and order. Functional equations belong to the same class, if their solutions are mutually expressible.

Problem 8: ***To establish relationships between decomposition of multiary functions and orthogonality.*** Using repetition-free compositions and other type of compositions of multiary functions, one can construct algorithms which construct tuples of orthogonal operations [4, 3, 20, 6],

Problem 9: ***Functional equations and continuous mathematics.*** Every composition of continuous functions is continuous. Therefore, all results obtained for quasigroups using composition are true for topological quasigroups. Since topological groups on real space with the standard topology are topologically isomorphic and continuous invertible functions are monotonic, then all solutions of functional equations on the set of all topological invertible real functions are expressed by monotonic functions and the addition of real numbers [16].

Problem 10: ***Functional equations and discrete mathematics.***

Preliminaries

A subset $\rho \subseteq Q^n := \underbrace{Q \times Q \times \ldots \times Q}_{n \text{ times}}$ is called an n-*ary relation* defined on Q. An n-ary relation is called *functional*, if the first $n-1$ element uniquely defines the n-th one, i.e.

$$(a_1, \ldots, a_{n-1}, b) \in \rho \ \wedge \ (a_1, \ldots, a_{n-1}, c) \in \rho \Rightarrow \ b = c.$$

In this case, the following relation is assumed to be true:

$$(a_1, \ldots, a_{n-1}, b) \in \rho \ \Leftrightarrow \ \rho(a_1, \ldots, a_{n-1}) = b.$$

If, in addition, for every $a_1, \ldots, a_{n-1} \in Q$ there exists $b \in Q$ such that $\rho(a_1, \ldots, a_{n-1}) = b$, then ρ is called an $(n-1)$-*ary operation* and it can be considered as a mapping of Q^{n-1} to Q.

So, in particular, a relation ρ is invertible or a *quasigroup relation*, or a *quasigroup operation* if in a formula $(x_1, \ldots, x_{n-1}, x_n) \in \rho$ arbitrary values of arbitrary $n-1$ variables uniquely define a value of the n-th variable. Evidently, this property is invariant under arbitrary permutation of the variables x_1, ..., x_{n-1}, x_n.

Let $\Omega_n(Q)$ be the set of all n-ary operations defined on Q. n associative superpositions $\underset{1}{\oplus}, \ldots, \underset{n}{\oplus}$ are defined on $\Omega_n(Q)$ by

$$\left(f \underset{i}{\oplus} g\right)(x_1, \ldots, x_n) := f(x_1, \ldots, x_{i-1}, g(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n), \quad i = 1, \ldots, n. \tag{2}$$

Monoid $(\Omega_n(Q); \underset{i}{\oplus})$ is called i-*th symmetric monoid* of n-ary operations. An n-ary operation is called i-*invertible*, if it is invertible in the i-th symmetric monoid, i.e., their exists an operation ${}^{[i]}f$, called an i-*th inverse operation*, such that

$$f \underset{i}{\oplus} {}^{[i]}f = e_i, \qquad {}^{[i]}f \underset{i}{\oplus} f = e_i, \tag{3}$$

where $e_i(x_1, \ldots, x_n) := x_i$ and e_i is called i-*th selector*. So, f is i-invertible, if the following identities (we will call them *primary identities*)

$$\begin{aligned} &f(x_1, \ldots, x_{i-1}, {}^{[i]}f(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n) = x_i \\ &{}^{[i]}f(x_1, \ldots, x_{i-1}, f(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n) = x_i. \end{aligned} \tag{4}$$

hold for some operation ${}^{[i]}f$. An n-ary operation is called *invertible*, if it is invertible in every symmetric monoid of n-ary operations:

$$(\Omega_n(Q); \underset{1}{\oplus}), \ \ \ldots, \ \ (\Omega_n(Q); \underset{n}{\oplus}).$$

In other words, f is invertible, if it has i-inverse operation for every $i = 1, \ldots, n$. Note, the full sequence ${}^{[1]}f$, ..., ${}^{[n]}f$ of inverse operations is uniquely defined.

An n-ary operation f is called an i-*quasigroup operation*, if for every a_1, ..., a_n, b the equation

$$f(a_1, \ldots, a_{i-1}, x, a_{n+1}, \ldots, a_n) = b \tag{5}$$

has a unique solution in Q. An assigning to every $(a_1, \ldots, a_{i-1}, b, a_{n+1}, \ldots, a_n)$ the solution of (5) is an n-ary operation. It is easy to verify that this operation is exactly i-inverse to f, so 'quasigroup operation' and 'inverse operation' are the same concept.

An algebra $(Q; f, {}^{[1]}f, \ldots, {}^{[n]}f)$ is called an n-*ary quasigroup*, if f is an invertible operation and ${}^{[1]}f$, ..., ${}^{[n]}f$ are its full sequence of inverse operations. Then class of all n-ary quasigroups forms a variety which is defined by primary identities (4).

Actions. Let S_n denote the symmetric group of the degree n, namely the group of all permutations of the set $\overline{1,n} := \{1, 2, \dots, n\}$ and let ι denote the identical transformation of an arbitrary set.

Let S_n act on a set K, i.e. for all $k \in K$ and for all $\sigma \in S_3$ the element ${}^{\sigma}k$ belongs to K and equalities

$$^{\sigma}(^{\tau}k) = {}^{\sigma\tau}k, \qquad {}^{\iota}k = k$$

hold. We adduce some statements which immediately follow from the well-known results of the group action theory. For this purpose, we introduce some special names and notations.

The relation $\sim$ being defined on K by

$$k \sim m \;\Leftrightarrow\; (\exists\sigma \in S_n)\;\; {}^{\sigma}k = m$$

is called *parastrophic*. An orbit of an element $k \in K$ will be called a *truss* of k and denoted by $\mathrm{Tr}(k)$; stabilizer group of k will be called a *parastrophic symmetry group* of k and denoted by $\mathrm{Ps}(k)$; a stabilizer group of $\mathrm{Ps}(k)$ under the conjugate action will be called a *normalizer group* of k or a *1-normalizer group* of k and will be denoted by $\mathrm{Norm}_1(1)$; the set of points fixed by g is called a *kernel* of σ and denoted by K^g. Hence,

$$\begin{aligned}
&\mathrm{Tr}(k) := \{m \in K \mid (\exists\sigma \in G)\, m = {}^{\sigma}k\}, && \mathrm{Ps}(k) := \{\sigma \mid {}^{\sigma}k = k\},\\
&\mathrm{Norm}_1(k) := \{\sigma \mid \mathrm{Ps}(k)\sigma = \sigma\mathrm{Ps}(k)\}, && K^{\sigma} := \{k \in K \mid {}^{\sigma}k = k\},\\
&\mathrm{Norm}_i(k) := \{\sigma \mid \mathrm{Norm}_{i-1}(k)\sigma = \sigma\mathrm{Norm}_{i-1}(k)\}, && i = 1, 2, \dots,\\
&\mathrm{Norm}_0(k) := \mathrm{Ps}(k).
\end{aligned}$$

In other words,

- a *truss* $\mathrm{Tr}(k)$ of k is its orbit under the action, i.e., the set of all elements which are parastrophic to k;
- a *parastrophic symmetry group* $\mathrm{Ps}(k)$ of k is its stabilizer group, i.e., the set of all parastrophisms which does not change the element k;
- a *normalizer group* $\mathrm{Norm}(k) = \mathrm{Norm}_1(k)$ of k is a set of all parastrophisms which does not change its parastrophic symmetry group under the conjugate action;
- a i*-th normalizer group* $\mathrm{Norm}_i(k)$ of k is a set of all parastrophisms which does not change its $(i-1)$-th normalizer group $\mathrm{Norm}_{i-1}(k)$ under the conjugate action.

Theorem 1. *Let n be a natural number and the group S_n act on a set K. Then the following properties are true:*

1. *$\sim$ is an equivalence relation on K and $\mathrm{Tr}(k)$ is a block of the corresponding partition of K which contains the element k;*
2. *a parastrophic symmetry group $\mathrm{Ps}(k)$ and normalizer groups $\mathrm{Norm}_i(k)$ of an arbitrary element $k \in K$ are subgroups of S_n and there exists s such that*
$$\mathrm{Ps}(k) \trianglelefteq \mathrm{Norm}_1(k) \trianglelefteq \mathrm{Norm}_2(k) \trianglelefteq \cdots \trianglelefteq \mathrm{Norm}_s(k) \trianglelefteq S_n;$$
3. *parastrophic symmetry groups and normalizer groups of elements from the same orbit are conjugate:*
$$\mathrm{Ps}(^{\sigma}k) = \sigma(\mathrm{Ps}(k))\sigma^{-1}, \qquad \mathrm{Norm}_i(^{\sigma}k) = \sigma(\mathrm{Norm}_i(k))\sigma^{-1}, \quad i = 1, \dots, s;$$
4. *the set of all parastrophisms between k and ${}^{\sigma}k$ is equal to $\sigma(\mathrm{Ps}(k))$;*
5. *$|\mathrm{Tr}(k)| = (n+1)!/|\mathrm{Ps}(k)|$ is the number of all different elements which are parastrophic to k;*
6. *$(n+1)!/|\mathrm{Norm}_i(k)|$ is the number of all different normalizer groups $\mathrm{Norm}_{i-1}(k)$ of elements from $\mathrm{Tr}(k)$;*
7. *$|\mathrm{Norm}(k)|/|\mathrm{Ps}(k)|$ is the number of all different elements in the truss of k which have the same parastrophic symmetry group;*

8. $|\mathrm{Norm}_i(k)|/|\mathrm{Ps}(k)|$ *is the number of all different elements in the truss of* k *which have the same normalizer symmetry group* $\mathrm{Norm}_{i-1}(k)$ *,* $i = 1, \dots, s$ *;*

9. *(Burnside's lemma)* $\frac{1}{(n+1)!}\sum_{\sigma\in S_3}|K^\sigma|$ *is the number of trusses, it is equal to the average number of points fixed per group element.*

Let H be a subgroup of S_n, then we will call an element k H*-symmetric* if $\mathrm{Ps}(k) \supseteq H$ and *strictly* H-symmetric if $\mathrm{Ps}(k) = H$. An element k will be called:

- *middle symmetric*, if $\mathrm{Ps}(k) \supseteq S_{n-1}$;
- *semisymmetric*, if $\mathrm{Ps}(k) \supseteq A_n$;
- *symmetric*, if $\mathrm{Ps}(k) \neq \{\iota\}$;
- *asymmetric*, if $\mathrm{Ps}(k) = \{\iota\}$;
- *totally symmetric*, if $\mathrm{Ps}(k) = S_n$.

If $n = 2$ we have full system of names, namely an element k is called

	totally symmetric,	if $\mathrm{Ps}(k) = S_3$;	$\lvert\mathrm{Tr}(k)\rvert = 1$;
	semisymmetric,	if $\mathrm{Ps}(k) \supseteq A_3$;	$\lvert\mathrm{Tr}(k)\rvert = 1, 2$;
One-sided symmetry	*middle symmetric*,	if $\mathrm{Ps}(k) \supseteq \{\iota, s\}$;	$\lvert\mathrm{Tr}(k)\rvert = 1, 3$;
	left symmetric,	if $\mathrm{Ps}(k) \supseteq \{\iota, r\}$;	$\lvert\mathrm{Tr}(k)\rvert = 1, 3$;
	right symmetric,	if $\mathrm{Ps}(k) \supseteq \{\iota, \ell\}$;	$\lvert\mathrm{Tr}(k)\rvert = 1, 3$;
	asymmetric,	if $\mathrm{Ps}(k) = \{\iota\}$	$\lvert\mathrm{Tr}(k)\rvert = 6$.

Corollary 1. *Let* H *be a normal subgroup of* S_n*. Then every element from* K *being parastrophic to an* H*-symmetric element is* H*-symmetric; all their normalizer groups coincide with* S_n*; and the number of these elements is equal to* $|\mathrm{Tr}(k)| = (n+1)!/|H|$.

For example, let k be a strictly semisymmetric element, that is $\mathrm{Ps}(k) = A_n$. Since A_n is a normal subgroup of S_n, then $\mathrm{Norm}(k) = S_n$. That is why the truss $\mathrm{Tr}(k)$ has two elements k and ${}^{(12)}k$, and A_n is the parastrophic symmetry group for both k and ${}^{(12)}k$. If k is asymmetric, i.e. $\mathrm{Ps}(k) = \{\iota\}$, then $\mathrm{Tr}(k)$ contains $(n+1)!$ different elements and all of them are asymmetric.

Example 1. *Let* S_4 *act on a set* M *and suppose*

$$K_4 := \{\iota, (12)(34), (13)(24), (14)(23)\}$$

is the parastrophic symmetry group of an element $k \in M$*. Since the group* K_4 *is normal in* S_4*, then there are 4 different elements in the orbit* $\mathrm{Tr}(k)$ *and the parastrophic symmetry group of each of these elements is equal to* K_4*.*

Some main actions of S_n

Relations and reloids

An action of S_n **on a set of** n**-ary relations.** Let Q be a set that is called a *carrier* or an *underlying set* and ρ be a subset of $Q^n := \underbrace{Q \times Q \times \cdots \times Q}_{n \text{ times}}$, then ρ is called an n*-ary relation* on Q. Denote

$$^{\sigma}\rho := \{(a_{1\sigma}, \dots, a_{n\sigma}) \mid (a_1, \dots, a_n) \in \rho\}. \tag{6}$$

Theorem 2. *An assignment* $(\sigma, \rho) \mapsto {}^{\sigma}\rho$ *defines an action of* S_n *on the set of all* n*-ary relations defined on* Q*.*

Proof. Let ρ be an arbitrary n-ary relation on Q and $\sigma, \tau \in S_n$, then

$$^{\tau}({}^{\sigma}\rho) = {}^{\tau}\{(a_{1\sigma}, \dots, a_{n\sigma}) \mid (a_1, \dots, a_n) \in \rho\} = \{(a_{1\tau\sigma}, \dots, a_{n\tau\sigma}) \mid (a_1, \dots, a_n) \in \rho\} = {}^{\tau\sigma}\rho.$$

□

An action of S_n on a set of n-ary reloids. Let $\mathcal{R} := (Q;\rho)$ be an n-ary reloid, i.e., Q is a set and $\rho \subseteq Q^n$. A pair ${}^{\sigma}\mathcal{R} := (Q;{}^{\sigma}\rho)$ is called a *σ-parastrophe* of $\mathcal{R}$.

Theorem 2. implies the following statement: "*A mapping $(\sigma,\mathcal{R}) \mapsto {}^{\sigma}\mathcal{R}$ defines an action of S_n on the set of reloids which are parastrophic to $\mathcal{R}$*".

An action of S_n on some sets of classes of n-ary reloids. Let $\mathfrak{R}$ be a class of n-ary reloids and ${}^{\sigma}\mathfrak{R}$ be a class of all reloids every of which is a σ-parastrophe of a reloid from $\mathfrak{R}$. A mapping $(\sigma,\mathfrak{R}) \mapsto {}^{\sigma}\mathfrak{R}$ defines an action of S_n on the set of reloid classes that are parastrophic to $\mathfrak{R}$.

An action of S_n on propositions from the language of a class of n-ary reloids. Let $\mathfrak{R}$ be a class of n-ary reloids and $\mathcal{L}$ be its language. Then σ-parastrophe ${}^{\sigma}P$ of a proposition P is a proposition obtained from P by replacing all appearances of the relation symbol ρ with its σ^{-1}-parastrophe ${}^{\sigma^{-1}}\rho$.

If we replace in a proposition P the symbol ρ with its σ^{-1}-parastrophe ${}^{\sigma^{-1}}\rho$, next in the obtained proposition ${}^{\sigma}P$ we replace the symbol ρ with its τ^{-1}-parastrophe ${}^{\tau^{-1}}\rho$, then we obtain a proposition ${}^{\tau}({}^{\sigma}P)$. It is the same when we replace the symbol ρ with ${}^{\sigma^{-1}}({}^{\tau^{-1}}\rho) = {}^{(\tau\sigma)^{-1}}\rho$ in the proposition P. Consequently, ${}^{\tau}({}^{\sigma}P) = {}^{\tau\sigma}P$.

Thus, a mapping $(\sigma,P) \mapsto {}^{\sigma}P$ defines an action of S_n on the set of all propositions from $\mathcal{L}$.

An action of S_n on concepts. Let a concept k be defined by a proposition P. A concept ${}^{\sigma}k$ which is defined by the proposition ${}^{\sigma}P$ is called a σ-parastrophe of k. It is easy to verify that the assigning $(\sigma,k) \to {}^{\sigma}k$ is an action of S_n on the set of all concepts.

An action of S_n on classes of n-ary reloids. Let $\mathfrak{A}$ be a class of n-ary reloids and ${}^{\sigma}\mathfrak{A}$ denote a class of all σ-parastrophes of reloids from the class $\mathfrak{A}$. It is easy to see that

$$ {}^{\tau}({}^{\sigma}\mathfrak{A}) = {}^{\tau\sigma}\mathfrak{A} \quad \text{for all } \tau,\sigma \in S_n. $$

The relationship implies that an assigning $(\sigma,\mathfrak{A}) \to {}^{\sigma}\mathfrak{A}$ is an action of the group S_n on the truss

$$ \mathrm{Tr}(\mathfrak{A}) := \{{}^{\sigma}\mathfrak{A} \mid \sigma \in S_n\}. $$

Theorem 3. *A proposition P is true in a class of reloids $\mathfrak{A}$ if and only if ${}^{\sigma}P$ is true in the class ${}^{\sigma}\mathfrak{A}$.*

Proof. Let $(Q;\rho)$ be arbitrary reloid from $\mathfrak{A}$ and $P(\rho)$ be a true proposition in $\mathfrak{A}$. Since $\rho = {}^{\sigma^{-1}}({}^{\sigma}\rho) = {}^{\sigma^{-1}}\theta$, where $\theta := {}^{\sigma}\rho$. Then $P({}^{\sigma^{-1}}\theta)$ is true proposition in ${}^{\sigma}\mathfrak{A}$, i.e., ${}^{\sigma}P$ is true proposition in ${}^{\sigma}\mathfrak{A}$. □

Corollary 2. *Let P be true in a class of reloids $\mathfrak{A}$ and let $\mathrm{Ps}(\mathfrak{A})$ be the group of parastrophic symmetries of $\mathfrak{A}$, then ${}^{\sigma}P$ is true in $\mathfrak{A}$ for all $\sigma \in \mathrm{Ps}(\mathfrak{A})$.*

Identities. Two identities are called

1. *equivalent*, if they define the same variety;
2. *primaryly equivalent*, if one of them can be obtained from the other by a composition of primary transformations (primaryly equivalent identities are equivalent);
3. *σ-parastrophic*, if one of them can be obtained from the other by σ-parastrophic transformation;
4. *parastrophic*, if they are σ-parastrophic for some $\sigma \in S_3$;
5. *σ-parastrophically equivalent*, if they define σ-parastrophic varieties (according to Theorem 3., σ-parastrophically equivalent identities define σ-parastrophic varieties);
6. *parastrophically equivalent*, if they are σ-parastrophically equivalent for some $\sigma \in S_3$;
7. *σ-parastrophically primary equivalent*, if one of them can be obtained from the other by a composition of primary transformations and σ_1, σ_2, …, σ_k parastrophic transformations such that $\sigma_1\sigma_2\dots\sigma_k = \sigma$ for some $k \in \mathbb{N}$;
8. *parastrophically primary equivalent*, if they are σ-parastrophically primary equivalent for some $\sigma \in S_3$.

Functions and groupoids

Recall that an n-ary function is a mapping $f:\ Q^n \mapsto Q$. From another point of view, f can be considered as $(n+1)$-ary relation:

$$(x_1,\dots,x_n,x_{n+1}) \in f :\Leftrightarrow\ f(x_1,\dots,x_n)=x_{n+1}.$$

A property which select functions among relations is the following:

$$(x_1,\dots,x_n,y)\in f\ \ \wedge\ \ (x_1,\dots,x_n,z)\in f \Rightarrow\ y=z.$$

It is called *functional property.*

For every $i=1,\dots,n$ a superposition $\underset{i}{\oplus}$ of two relations is defined as follows:

$$(a_1,\dots,a_{i-1},b,a_i,\dots,a_{n-1},c)\in \varphi \underset{i}{\oplus} \rho :\Leftrightarrow$$
$$\Big(\exists a\Big)(a_1,\dots,a_{i-1},b,a_i,\dots,a_{n-1},a)\in\varphi\ \ \wedge\ \ (a_1,\dots,a_{i-1},a,a_i,\dots,a_{n-1},c)\in\rho. \qquad (7)$$

Let $[i]:=(i,n+1)$, then a relation ρ will be called *i-invertible*, if

$${}^{[i]}\rho \underset{i}{\oplus} \rho = \rho \underset{i}{\oplus} {}^{[i]}\rho = e_i, \quad i=1,2\dots,n,$$

where $e_i:=\{(a_1,\dots,a_{i-1},a,a_i,\dots,a_{n-1},a)\mid a_1,\dots,a_{i-1},a,a_i,\dots,a_{n-1}\in Q\}$ and is called *i-th selector.* An $(n+1)$-ary relation is called *invertible* if it is i-invertible for all $i=1,2\dots,n$.

Proposition 1. *Let $\mathcal{R}_{n+1}$ be the set of all $(n+1)$-ary relations defined on a set Q. For every $i=1,\dots,n$ the algebra $(\mathcal{R}_{n+1};\underset{i}{\oplus},e_i)$ is a monoid whose group of invertible elements is the set of all i-invertible relations; besides every i-invertible $(n+1)$-ary relation f is an i-invertible n-ary function and ${}^{[i]}f$ is its inverse in the monoid (it is called i-*th division *of f).*

Proof. Let a relation f be i-invertible for some $i=1,\dots,n+1$. i-invertibility implies $f\underset{i}{\oplus}{}^{[i]}f=e_i$. Let a_1, ..., a_{n-1}, a, b, c be arbitrary elements from the underlying set Q such that

$$(a_1,\dots,a_{i-1},a,a_i,\dots,a_{n-1},b)\in f\ \ \wedge\ \ (a_1,\dots,a_{i-1},a,a_i,\dots,a_{n-1},c)\in f.$$

According to definition of a parastrophe of a relation we have

$$(a_1,\dots,a_{i-1},b,a_i,\dots,a_{n-1},a)\in {}^{[i]}f\ \ \wedge\ \ (a_1,\dots,a_{i-1},a,a_i,\dots,a_{n-1},c)\in f.$$

It means that $(a_1,\dots,a_{i-1},b,a_i,\dots,a_{n-1},c)\in f\underset{i}{\oplus}{}^{[i]}f=e_i$, so $b=c$. Therefore, f is an n-ary operation.

It is easy to verify associativity of $\underset{i}{\oplus}$ and that ${}^{[i]}f(a_1,\dots,a_{i-1},b,a_i,\dots,a_{n-1})$ is the unique solution of the equation

$$f(a_1,\dots,a_{i-1},x,a_i,\dots,a_{n-1})=b.$$

□

An action of S_n on a set of n-ary operations. Let Q be a set and f be an n-ary operation, i.e., $f:\ Q^n\mapsto Q$. Define ${}^{\sigma}\!f$ as follows

$${}^{\sigma}\!f(x_1,\dots,x_n):=f(x_{1\sigma^{-1}},\dots,x_{n\sigma^{-1}}), \qquad \sigma\in S_n. \qquad (8)$$

A mapping $(\sigma,f)\mapsto{}^{\sigma}\!f$ defines an action of S_n on the set of all n-ary functions defined on Q. In this case, i.e. $\sigma\in S_n$, the operation ${}^{\sigma}\!f$ is called a *principal* parastrophe of f.

It is easy to see that this action is a special case of the action of S_n as a subgroup of S_{n+1} on the set of all $(n+1)$-ary relations with the $(n+1)$-functional property.

An action of S_{n+1} on a set of n-ary invertible operations (quasigroup operations). Let Q be a set. An n-ary invertible operation f is called a *quasigroup operation* as well.

The definition (8) for a σ-parastrophe ${}^{\sigma}\!f$ of an n-ary operation f can be rewritten as follows:

$${}^{\sigma}\!f(x_{1\sigma},\dots,x_{n\sigma})=x_{(n+1)\sigma} :\Leftrightarrow\ f(x_1,\dots,x_n)=x_{n+1}, \quad \sigma\in S_{n+1}. \qquad (9)$$

A mapping $(\sigma,f)\mapsto{}^{\sigma}\!f$ is an action of the group S_{n+1} on the set Δ_n of all invertible n-ary operations defined on Q, since it is a partial case of the action S_{n+1} on $\mathcal{R}_{n+1}$.

Action of S_{n+1} on n-ary quasigroups. A groupoid ${}^{\sigma}\mathcal{A} := (A; \overset{\sigma}{\cdot})$ is called a σ*-parastrophe* of a quasigroup $\mathcal{A} := (A; \cdot)$. $(\sigma; \mathcal{A}) \mapsto {}^{\sigma}\mathcal{A}$ is an action of S_{n+1} on $\{{}^{\tau}\mathcal{A} \mid \tau \in S_{n+1}\} = \mathrm{Tr}(\mathcal{A})$.

If $n = 2$ we have a well-known classification. Namely, a binary quasigroup $\mathcal{A} := (A; \cdot)$ is

	totally symmetric,	if $\mathrm{Ps}(\mathcal{A}) = S_3$;	$xy = yx$, $x \cdot xy = y$;
One-sided symmetric	*semisymmetric*,	if $\mathrm{Ps}(\mathcal{A}) \supseteq A_3$;	$x \cdot yx = y$;
	commutative,	if $\mathrm{Ps}(\mathcal{A}) \supseteq \{\iota, s\}$;	$xy = yx$;
	left symmetric,	if $\mathrm{Ps}(\mathcal{A}) \supseteq \{\iota, r\}$;	$x \cdot xy = y$;
	right symmetric,	if $\mathrm{Ps}(\mathcal{A}) \supseteq \{\iota, \ell\}$;	$xy \cdot y = x$;
	asymmetric,	if $\mathrm{Ps}(\mathcal{A}) = \{\iota\}$	

An action of S_{n+1} on classes of n-ary quasigroups. Let $\mathfrak{A}$ be a class of n-ary quasigroups and ${}^{\sigma}\mathfrak{A}$ denote a class of all σ-parastrophes of quasigroups from the class $\mathfrak{A}$. It is easy to see that

$$^{\tau}\left({}^{\sigma}\mathfrak{A}\right) = {}^{\tau\sigma}\mathfrak{A} \quad \text{for all } \tau, \sigma \in S_{n+1}.$$

It implies that an assigning $(\sigma, \mathfrak{A}) \to {}^{\sigma}\mathfrak{A}$ is an action of the group S_{n+1} on the truss $\mathrm{Tr}(\mathfrak{A})$.

Bunches

A parastrophically closed semi-lattice of classes of quasigroups will be called a *bunch.* A *bunch of a reloid* $\mathfrak{A}$ is said to be a set of all parastrophes of $\mathfrak{A}$ and all their finite intersections.

Consider a bunch of a class $\mathfrak{A}$ of binary quasigroups. It consists of the following classes:

1. the set of all parastrophes of the class $\mathfrak{A}$, i.e. the truss of $\mathfrak{A}$:
$$\mathrm{Tr}\mathfrak{A} = \{\mathfrak{A},\ {}^{s}\mathfrak{A},\ {}^{\ell}\mathfrak{A},\ {}^{r}\mathfrak{A},\ {}^{s\ell}\mathfrak{A},\ {}^{sr}\mathfrak{A}\};$$
2. the set of all pairwise intersections of the classes from $\mathrm{Tr}\mathfrak{A}$, i.e.
$$\{{}^{\tau}\mathfrak{A} \cap {}^{\nu}\mathfrak{A} \mid \tau, \nu \in S_3\};$$
3. the set of all triple-wise intersections of the classes from $\mathrm{Tr}\mathfrak{A}$, i.e.
$$\{{}^{\nu_1}\mathfrak{A} \cap {}^{\nu_2}\mathfrak{A} \cap {}^{\nu_3}\mathfrak{A} \mid \nu_1, \nu_2, \nu_3 \in S_3\};$$
4. the set of all quadruple-wise intersections of the classes from $\mathrm{Tr}\mathfrak{A}$, i.e.
$$\{{}^{\nu_1}\mathfrak{A} \cap {}^{\nu_2}\mathfrak{A} \cap {}^{\nu_3}\mathfrak{A} \cap {}^{\nu_4}\mathfrak{A} \mid \nu_1, \nu_2, \nu_3, \nu_4 \in S_3\};$$
5. the set of all quintuple-wise intersections of the classes from $\mathrm{Tr}\mathfrak{A}$, i.e.
$$\{{}^{\nu_1}\mathfrak{A} \cap {}^{\nu_2}\mathfrak{A} \cap {}^{\nu_3}\mathfrak{A} \cap {}^{\nu_4}\mathfrak{A} \cap {}^{\nu_5}\mathfrak{A} \mid \nu_1, \nu_2, \nu_3, \nu_4, \nu_5 \in S_3\};$$
6. the intersections of all classes from $\mathrm{Tr}\mathfrak{A}$, i.e.
$$\mathfrak{A} \cap {}^{s}\mathfrak{A} \cap {}^{\ell}\mathfrak{A} \cap {}^{r}\mathfrak{A} \cap {}^{s\ell}\mathfrak{A} \cap {}^{sr}\mathfrak{A}.$$

The bunch of binary loops

Neutral elements and loops. Let $(Q; f)$ be an n-ary quasigroup. An element e of the quasigroup is said to be:

1. (i, j)*-neutral*, if
$$f(x_0, \ldots, x_{n-1}) = x_n,$$
where $x_i = x_j$ and $x_k = e$ for all k such that $i, j \neq k \in \overline{0, n}$. In this case, $(x_0, \ldots, x_n)$ is called a *defining sequence*;
2. *unilateral*, if it is (i, j)-neutral for some (i, j);
3. *neutral*, if it is (i, j)-neutral for all pairs (i, j) such that $0 \leqslant i, j \leqslant n - 1$;
4. *totally neutral*, if it is (i, j)-neutral for all pairs (i, j) such that $0 \leqslant i, j \leqslant n$.

If $(x_0, \ldots, x_n)$ is a defining sequence in a quasigroup, then $(x_{0\sigma}, \ldots, x_{n\sigma})$ is a defining sequence in its σ-parastrophe. So, if an element e is (i, j)-neutral in a quasigroup, then it is $(i\sigma^{-1}, j\sigma^{-1})$-neutral in its σ-parastrophe. Therefore, the concepts of 'unilateral element' and 'totally neutral element' are totally symmetric.

For example, neutral element in a Boolean group is totally neutral and each element in a ternary quasigroup $(Q; f)$ which is defined on a Boolean group $(Q; +)$ by $f(x, y, z) = x + y + z$, is neutral.

Neutrality for binary quasigroups. In binary case ($n = 2$) there are three defining sequences (e, x, x), (x, e, x), (x, x, e) of neutrality. So, in an arbitrary binary quasigroup $(Q; \cdot)$ one can define three types of neutrality: 1-neutral (*left neutral*) $e \cdot x = x$, 2-neutral (*right neutral*) $x \cdot e = x$ and 3-neutral (*middle neutral*) $x \cdot x = e$. A quasigroup with a unilaterally neutral element is called a *unilateral* or *one-sided loop*. A middle neutral element is also known as *unipotent element*.

To prove general properties of neutral elements we have to formulate a general definition which is a partial case of the definition for n-ary quasigroups. Namely, an element e of a quasigroup $\mathcal{A} := (Q; \cdot)$ is called:

1. i*-neutral* if the equality $x_1 \cdot x_2 = x_3$ is true, where $x_i = e$ and the other two variables coincide, and *unilaterally neutral*, if it is i-neutral for some $i = 1, 2, 3$;

2. $\{i, j\}$*-neutral* if it is i-neutral and j-neutral and *two-sided neutral*, if it is $\{i, j\}$*-neutral* for some $i, j = 1, 2, 3$;

3. *totally neutral*, if it is i-neutral for all $i = 1, 2, 3$.

Proposition 2. *If an element e is i-neutral in a loop $\mathcal{A}$, then it is $i\sigma^{-1}$-neutral in ${}^{\sigma}\!\mathcal{A}$ for all $i = 1, 2, 3$. Every parastrophe of a unilateral loop is a unilateral loop. Every unilateral loop $\mathcal{A}$ has exactly one neutral element. The element is neutral in all parastrophes of $\mathcal{A}$.*

Proof. Let e be an i-neutral element in a loop $\mathcal{A}$. It means that the equality $x_1 \cdot x_2 = x_3$ is true, where $x_i = e$ and two another variables are equal. This equality is equivalent to $x_{1\sigma} \overset{\sigma}{\cdot} x_{2\sigma} = x_{3\sigma}$ according to the definition of σ-parastrophe. Since $e = x_i = x_{(i\sigma^{-1})\sigma}$, then the element e is $i\sigma^{-1}$-th neutral in ${}^{\sigma}\!\mathcal{A}$.

Let an element e be i-neutral in $\mathcal{A}$ and e' be j-neutral in ${}^{\sigma}\!\mathcal{A}$, then e' is $j\sigma$-neutral in $\mathcal{A}$. If $j\sigma = i$, then $e = e'$ since $\mathcal{A}$ is a quasigroup. If $j\sigma \neq i$, we consider the permutation

$$\tau := \begin{pmatrix} 1 & 2 & 3 \\ i & j\sigma & k \end{pmatrix}, \quad \text{where } \{i, j\sigma, k\} = \{1, 2, 3\}.$$

Using just proved assertions, the element e is $i\tau^{-1}$-neutral and e' is $j\sigma\tau^{-1}$-neutral in ${}^{\tau}\!\mathcal{A}$. Since $i\tau^{-1} = 1$ and $j\sigma\tau^{-1} = 2$, then e is left neutral and e' is right neutral in ${}^{\tau}\!\mathcal{A}$, so $e = e'$. □

Thus, an element with a property of neutrality is unique and the same is true for all parastrophes. But it can be *one-sided*, if it has at least one of the properties of neutrality (left, right, middle), *two-sided* if it has at least two of these properties or *three-sided* (*totally neutral*) if it satisfies all properties of neutrality.

The following table shows what kind of neutrality an element has in parastrophes of a quasigroup $\mathcal{A}$, if it is left neutral or left-right neutral in $\mathcal{A}$.

	$\mathcal{A}$	${}^{s}\!\mathcal{A}$	${}^{\ell}\!\mathcal{A}$	${}^{r}\!\mathcal{A}$	${}^{sr}\!\mathcal{A}$	${}^{s\ell}\!\mathcal{A}$
one-sided	1-neutral left	2-neutral right	3-neutral middle	1-neutral left	2-neutral right	3-neutral middle
two-sided	12-neutral left-right neutral		13-neutral left-middle neutral		23-neutral right-middle neutral	
three-sided	123-neutral, i.e. totally neutral					

Therefore

Corollary 3. *There exist seven varieties of loops:*

The bunch of varieties of loops			
the varieties of one-sided loops	$\mathfrak{L}\ \ (= {}^{r}\mathfrak{L})$	${}^{s}\mathfrak{L}\ \ (= {}^{sr}\mathfrak{L})$	${}^{\ell}\mathfrak{L}\ \ (= {}^{s\ell}\mathfrak{L})$
	1-loops, i.e. left loops	*2-loops, i.e. right loops*	*3-loops, i.e. middle loops*
	$x \overset{\ell}{\cdot} x = y \overset{\ell}{\cdot} y$	$x \overset{r}{\cdot} x = y \overset{r}{\cdot} y$	$x \cdot x = y \cdot y$
the varieties of two-sided loops	$\mathfrak{L} \cap {}^{s}\mathfrak{L}$	$\mathfrak{L} \cap {}^{\ell}\mathfrak{L}$	${}^{s}\mathfrak{L} \cap {}^{\ell}\mathfrak{L}$
	12-loops, i.e. left-right loops	*13-loops, i.e. left-middle loops*	*23-loops, i.e. right-middle loops*
	$x \overset{\ell}{\cdot} x = y \overset{r}{\cdot} y$	$x \overset{\ell}{\cdot} x = y \cdot y$	$x \overset{r}{\cdot} x = y \cdot y$
the variety of three-sided loops	$\mathfrak{L} \cap {}^{s}\mathfrak{L} \cap {}^{\ell}\mathfrak{L}$		
	total loops, i.e. unipotent loops		
	$x^2 y = y, \ \ yx^2 = y$		

Note that a left-right neutral element are traditionally called neutral or an identity element.

The varieties in the same row are parastrophic, therefore their parastrophic symmetry groups are conjugate according to p. 3 of Theorem 1.. Indeed,

$$^{\ell}(\mathfrak{L}\cap {}^{s}\mathfrak{L}) = {}^{\ell}\mathfrak{L}\cap {}^{\ell s}\mathfrak{L} = {}^{\ell}\mathfrak{L}\cap {}^{sr}\mathfrak{L} = {}^{\ell}\mathfrak{L}\cap {}^{s}\mathfrak{L} = {}^{s}\mathfrak{L}\cap {}^{\ell}\mathfrak{L},$$

$$^{r}(\mathfrak{L}\cap {}^{s}\mathfrak{L}) = {}^{r}\mathfrak{L}\cap {}^{rs}\mathfrak{L} = \mathfrak{L}\cap {}^{s\ell}\mathfrak{L} = \mathfrak{L}\cap {}^{\ell}\mathfrak{L}.$$

It is easy to varify that

$$^{\sigma}(\mathfrak{L}\cap {}^{s}\mathfrak{L}\cap {}^{\ell}\mathfrak{L}) = \mathfrak{L}\cap {}^{s}\mathfrak{L}\cap {}^{\ell}\mathfrak{L}, \quad \text{for all } \sigma\in S_3.$$

All of these concepts are pairwise parastrophic. Indeed, according to Proposition 2., 12-neutral element in a quasigroup $\mathcal{A}$ is 13-neutral element in ${}^{r}\mathcal{A}$ and 23-neutral element in ${}^{\ell}\mathcal{A}$.

An example of parastrophic propositions. Consider a well-known proposition P: *'Every quasigroup is isotopic to a left-right loop (=loop)'.* And let us find all its parastrophes. The concept of '*quasigroup*' is totally symmetric because every parastrophe of a quasigroup is a quasigroup; if quasigroups are isotopic then their σ-parastrophes are isotopic as well, so isotopy is totally symmetric. As we have shown above, the concept of two-sided loops is middle symmetric. Thus, we have three different parastrophes of P:

- ${}^{\ell}P$: '*Every quasigroup is isotopic to a right-middle loop*';
- ${}^{r}P$: '*Every quasigroup is isotopic to a left-middle loop*'.

Thus the following theorem is true.

Theorem 4. *1. Every quasigroup is isotopic to a left-right loop (=loop):*

$$x\circ y = R_a^{-1}(x)\cdot L_b^{-1}(y), \qquad e = ba = R_a(b) = L_b(a);$$

2. Every quasigroup is isotopic to a right-middle loop:

$$x\circ y = R_b^{-1}\Big(x\cdot M_c(y)\Big), \qquad e = R_b^{-1}(c) = M_c^{-1}(b);$$

3. Every quasigroup is isotopic to a left-middle loop:

$$x\circ y = L_a^{-1}\Big(M_c(x)\cdot y\Big), \qquad e = M_c^{-1}(a) = L_a^{-1}(c);$$

4. a quasigroup $(Q;\cdot)$ is isotopic to a unipotent loop if and only if it has elements a,b such that $M_{ab} = L_b^{-1}R_a$:

$$x\circ y = R_a^{-1}(x)\cdot L_b^{-1}(y).$$

Examples of bunches. Here we cite some of well-known trusses and bunches of quasigroup varieties.

Example 2. *The bunch of all loops consists of seven varieties (see Corollary 3.).*

Example 3. *The bunch of distributive quasigroups consists of one totally symmetric variety.*

Proof. Because every parastrophe of a distributive quasigroup is also distributive [14]. □

Example 4. *The bunch of all groups consists of four varieties:*

- *variety of groups,* $xy\cdot z = x\cdot yz$;
- *variety of left division of groups,* $xy\cdot zy = xz$;
- *variety of right division of groups,* $xy\cdot xz = yz$;
- *variety of Boolean groups* $xy\cdot yz = xz$.

Proof. Let $\mathfrak{G}$ be the variety of all groups. The class ${}^{\ell}\mathfrak{G}$ of all left divisions is the class of all quasigroups $(Q;\overset{\ell}{\cdot})$ such that $(Q;\cdot)$ belongs to $\mathfrak{G}$. According to Theorem 3., the class ${}^{\ell}\mathfrak{G}$ is described by ℓ-parastrophe of associativity. Since $\ell^2=\iota$, then it is the identity

$$(x\overset{\ell}{\cdot}y)\overset{\ell}{\cdot}z=x\overset{\ell}{\cdot}(y\overset{\ell}{\cdot}z).$$

Let us use the primary identities. Denote $v:=y\overset{\ell}{\cdot}z$, then $y=vz$; and $u:=(x\overset{\ell}{\cdot}y)\overset{\ell}{\cdot}z$ implies $x=uz\cdot y=uz\cdot vz$. Therefore, the identity can be written as

$$u=(uz\cdot vz)\overset{\ell}{\cdot}v.$$

Applying a primary identity, we obtain $uz\cdot vz=uv$. Analogously one can show that the class ${}^{r}\mathfrak{G}$ can be described by $xy\cdot xz=yz$. According to Theorem 3., the variety ${}^{s}\mathfrak{G}$ is described by

$$(x\overset{s}{\cdot}y)\overset{s}{\cdot}z=x\overset{s}{\cdot}(y\overset{s}{\cdot}z).$$

As $x\overset{s}{\cdot}y=y\cdot x$, this is the identity of associativity. Therefore, ${}^{s}\mathfrak{G}=\mathfrak{G}$. It means $Ps(\mathfrak{G})\supseteq\{\iota,s\}$. Let $(\mathbb{Z};+)$ be the group of integers. Then the quasigroups $(\mathbb{Z};+)$, $(\mathbb{Z};\overset{\ell}{+})$, $(\mathbb{Z};\overset{r}{+})$ are pairwise different because the operations $(+)$, $(\overset{\ell}{\cdot})$, $(\overset{r}{\cdot})$ are different. These statements follow from the equalities

$$2+3=5,\qquad 2\overset{\ell}{+}3=2-3=-1,\qquad 2\overset{r}{+}3=3-2=1.$$

Thus, ${}^{s}\mathfrak{G}=\mathfrak{G}$, ${}^{\ell}\mathfrak{G}={}^{sr}\mathfrak{G}$, ${}^{r}\mathfrak{G}={}^{s\ell}\mathfrak{G}$ are pairwise different varieties and $Ps(\mathfrak{G})=\{\iota,s\}$. An intersection of any two of them gives the variety of Boolean groups. □

Identities and functional equations

The concept of 'identity' can be divided into two parts:

1. a *proposition*, for example, in the real number group $(\mathbb{R};+)$, the following identity is true
$$(\forall x)(\forall y)(\forall z)\ \ (x+y)+z=x+(y+z); \tag{10}$$
2. a *predicate*, for example, the class of all semigroups is defined by the following identity
$$(\forall x)(\forall y)(\forall z)\ \ (x+y)+z=x+(y+z); \tag{11}$$

In 1) the symbol $+$ denotes a fixed operation, namely addition of the real numbers, but in 2) the symbol $+$ denotes a functional variable. We need to distinguish the concepts. Therefore, we keep the name 'identity' only for (10), and (11) will be called a functional equation. For giving exact definitions we remember the concept of a term.

Terms and words. Let Q be a set which will be called *carrier* or *underlying set*. Let

- $\mathcal{Q}:=\{a,b,c,a_1,\dots\}$ be a set of fixed elements from Q (individual constant);
- $\mathcal{F}:=\{f,g,h,f_1,f_2,\dots\}$ a set of functional symbols which denote one and only one operation defined on Q (functional constant);
- $\mathcal{X}:=\{x,y,x_1,x_2,\dots\}$ a set of individual variables representing the elements from Q;
- $\mathfrak{F}:=\{F,F_1,F_1,\dots\}$ be a set of functional variables.

Definition of a *term*:

1. every variable from $\mathcal{X}$ and every individual constant from $\mathcal{Q}$ are tems;
2. if $f\in\mathcal{F}$ is an n-ary function, $F\in\mathfrak{F}$ an n-ary functional variable and T_1, …, T_n are terms, then $f(T_1,\dots,T_n)$, $F(T_1,\dots,T_n)$ are terms;
3. no terms exist other than those implied by the previous rules.

A term is called a *word*, if it has no functional variable. Let T be a term then $[T]$ and $\langle T\rangle$ denote the sets of all individual and functional variables appearing in T respectively.

Let $[T_1]\cup[T_2]:=\{x_1,\dots,x_n\}$ and $\{F_1,F_2,\dots,F_k\}\subseteq\langle T_1\rangle\cup\langle T_2\rangle$, then the formula

$$(\forall F_1)(\forall F_2)\dots(\forall F_k)(\forall x_1)(\forall x_2)\dots(\forall x_n)(T_1=T_2) \tag{12}$$

is called a *universal (quantified) equality.* We will denote it without quantifiers.

Definition 1. *A universal equality (12) is called a* functional equation on Q *if it has at least one free functional variable, otherwise it is called an* identity *if it is true and a* contradiction *if it is false.*

For example, let F_1 be unary and F_3 be binary real functions defined everywhere, then

$$(\forall F_1)(\forall F_3)(\forall x)(\forall y)(\forall z)\Big(\big(F_1(x)+\sin x\big)+F_3(x,y)=F_1(x)+\big(\sin x+F_3(x,y)\big)\Big)$$

is an identity on real numbers. In this formula $(+)$ is binary and $\sin$ is unary functional constant.

Definition 2. *A functional equation is called* pure*, if it has neither functional constant nor individual constant.*

Definition 3. *A value of lexicographic sequence of all free functional variables of the given functional equation is called its* solution*, if the equation becomes an identity after substituting of the solution for functional variables.*

Pure functional equation can be considered on every carrier and on every carrier it has some set of solutions. So, a solution of a pure functional equation is a pair: a carrier and a sequence of functions defined on the carrier. Therefore, all solutions of a pure functional equation form a class of algebras. The class is called a variety and the functional equation is called an identity which describes the variety.

Definition 4. *A formula (12) is called a* universal quasigroup equality *if its both functional variables and functional constants present quasigroup operations.*

A primary quasigroup super-identity is a pure quasigroup identity which follows from the definition of an invertible operation and its parastrophes. For binary case, these identities are the following:

$$\begin{aligned} &{}^{\sigma}({}^{\tau}F)={}^{\sigma\tau}F, && {}^{s}F(x,y)=F(y,x),\\ &{}^{\ell}F(F(x,y),y)=x, && F({}^{\ell}F(x,y),y)=x,\\ &{}^{r}F(x,F(x,y))=y, && F(x,{}^{r}F(x,y))=y,\\ &{}^{s\ell}F(x,F(y,x))=y, && F({}^{s\ell}F(x,y),x)=y,\\ &{}^{sr}F(F(y,x),y)=x, && F(y,{}^{sr}F(x,y))=x. \end{aligned} \tag{13}$$

Definition 5. *Two functional equations are said to be* equivalent *on a carrier if they have the same set of solutions on the carrier. Two pure functional equations are called equivalent if they are equivalent on each carrier.*

Following Sade [11], an operation will be called *diagonal*, if $f(x;x)$ is a permutation of the carrier set. A binary functional variable will be called *diagonal*, if it presents diagonal operations.

Two functional equations are said to be *parastrophically primarily equivalent*, if one can be obtained from the other in a finite number of the following steps:

1) application of quasigroup superidentities (13);
2) changing sides of the equation;
3) renaming individual variables;
4) renaming functional variables.

Two functional equations are said to be *diagonaly parastrophic*, if one can be obtained from the other in a finite number of the following steps:

1) application of quasigroup superidentities (13);

2) changing sides of the equation;

3) renaming individual variables;

4) renaming functional variables;

5) replacing a sub-term $F(x;x)$ with $\delta_F(x)$, if F is a diagonal functional variable and vice versa.

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ПАРАСТРОФНА СИМЕТРІЯ В ТЕОРІЇ КВАЗІГРУП

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РЕЗЮМЕ

У цій статті виділено декілька проблем теорії квазігрупп. Особлива увага приділяється проблемі парастрофної симетрії. Дано означення парастрофних відношень на: відношеннях та операціях, многовидах квазігруп, твердженнях, квазігрупових тотожностях, поняттях в теорії квазігруп тощо. Як приклад, наведено парастрофно замкнутий набір понять односторонніх нейтральних елементів, а також відповідних многовидів.

Ключові слова: квазігрупа, лупа, симетрія, нейтральний елемент, парастроф, многовид, пучок, в'язка, тототожність, функційне рівняння.

ПАРАСТРОФНАЯ СИММЕТРИЯ В ТЕОРИИ КВАЗИГРУПП

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РЕЗЮМЕ

В этой статье выделено несколько проблем в теории квазигрупп. Особое внимание уделяется проблеме парастрофной симметрии: дано определение парастрофнии на: отношениях и операциях, многообразиях квазигрупп, утверждениех, квазигруповых тождествах, понятиях в теории квазигруп и прочее. В качестве примера, приведен парастрофно замкнутый набор понятий односторонних нейтральных элементов, а также соответствующие многообразия.

Ключевые слова: квазигруппа, лупа, симметрия, нейтральный елемент, парастроф, многообразие, пучок, вязка, тождество, функциональное уравнение.