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THE TRUSS CONTAINING THE VARIETY OF COMMUTATIVE MEDIAL QUASIGROUPS

The truss containing the variety of commutative medial quasigroups is under consideration. The identities in four variables describing the varieties of left symmetric and right symmetric medial quasigroups are found. Canonical decompositions of quasigroups of these varieties are given. Parastrophic symmetry of varieties is described.
Keywords: *qroup, quasigroup, identity, variety, parastrophe, isotope, medial, commutative, left symmetric, right symmetric.*

Introduction

It is well known that the classes of all commutative, left symmetric and right symmetric quasigroups are described by the identities

$$xy = yx, \qquad xy \cdot y = x, \qquad x \cdot xy = y \tag{1}$$

respectively. These identities are called commutativity, left symmetry and right symmetry. The last two identities are also called Sade's left "keys" law and Sade's right "keys" law [10]. Two arbitrary identities from (1) define the class of totally symmetric quasigroups. S.K. Stein [15] established parastrophic relationships among identities (1) and described corresponding varieties, therefore these varieties belong to the same truss. T. Popovych [9] and G.B. Belyavskaya [2] gave all sets of identities in two variables which are equivalent to each of the identities (1).

We will restrict our attention to studying of medial quasigroups. Every medial quasigroup, that is a quasigroup satisfying the identity

$$xy \cdot uv = xu \cdot yv,$$

is isotopic to some Abelian group, therefore it can be constructed from this Abelian group (it is a well-known Toyoda-Bruck theorem [1]). S.K. Stein [15] proved that all parastrophes of medial identity are primarily equivalent to mediality. It means that the variety of medial quasigroups is totally symmetric. M. Polonijo [8] found identities which are equivalent to commutative mediality, that is they define the variety of commutative medial quasigroups. A. Krapež [7] found canonical decomposition of such quasigroups.

F. Sokhatsky [13] described a symmetry concept for parastrophes of quasigroup varieties and their quasigroups. The notion of a truss introduced by F. Sokhatsky [14] as a set of all pairwise parastrophic varieties. If one variety from a truss is investigated, then the symmetry concept allow to formulate a lot of properties for the rest varieties of this truss without their separate investigation.

H. Krainichuk [3] established that according to symmetry concept, all group isotopes are distributed into six classes: totally symmetric, semisymmetric, commutative (middle symmetric), left symmetric, right symmetric and asymmetric. All listed classes form three totally symmetric trusses:

- the truss which is the variety of totally symmetric quasigroups;
- the truss which is the variety of semisymmetric quasigroups;
- the truss of asymmetric quasigroups which is not a variety

and one middle symmetric truss that consists of three parastrophic varieties of commutative, left symmetric and right symmetric quasigroups.

According to Krainichuk's classification [3], there are no semisymmetric and asymmetric trusses. But in the asymmetric class, different types of trusses and varieties could be selected. In particular, semisymmetric truss of semisymmetric varieties consisting of asymmetric quasigroups was selected in [4], [5] (however "skew symmetry" instead of "semisymmetry" was used). It is a solution of the problem formulated by F. Sokhatsky [12].

According to symmetry concept, medial quasigroups being Abelian group isotopes are distributed into six mentioned above classes. Totally symmetric medial quasigroups have only the type $x \cdot y = -x + a - y$.

In [20], the author selected the variety of semisymmetry-like medial quasigroups from the class of asymmetric quasigroups, where quasigroups have mutually inverse coefficients in their canonical decompositions. Also, its two subvarieties are selected. Identities in four variables defining these three varieties are found, corresponding trusses of varieties are described.

The variety of semisymmetric medial quasigroups and corresponding identities are characterized in [6]. In particular, this variety is semisymmetrically isotopic closure of Abelian group varieties.

The aim of the article is to describe the truss containing the variety of commutative medial quasigroups. This truss is middle symmetric, that is it consists of three varieties: the variety $\mathfrak{C}$ of commutative medial quasigroups, the variety ${}^{\ell}\mathfrak{C}$ of left symmetric medial quasigroups and the variety ${}^{r}\mathfrak{C}$ of right symmetric medial quasigroups. Therefore, we give series of identities determining varieties of left symmetric medial quasigroups (Proposition 4.) and right symmetric medial quasigroups (Proposition 5.). Canonical decompositions of quasigroups of these varieties are established (Theorem 7., 8.). The truss of varieties of commutative, left symmetric and right symmetric medial quasigroups is characterized (Theorem 6.).

Preliminaries

An algebra with three operations $(Q;\cdot,\overset{\ell}{\cdot},\overset{r}{\cdot})$ is a quasigroup [1], if the identities

$$(x\cdot y)\overset{\ell}{\cdot} y=x,\quad (x\overset{\ell}{\cdot} y)\cdot y=x,\quad x\overset{r}{\cdot}(x\cdot y)=y,\quad x\cdot(x\overset{r}{\cdot} y)=y \tag{2}$$

hold, where $(\overset{\ell}{\cdot})$ and $(\overset{r}{\cdot})$ are left and right divisions of $(\cdot)$. They are defined by:

$$x\cdot y=z\Leftrightarrow x\overset{r}{\cdot} z=y\Leftrightarrow z\overset{\ell}{\cdot} y=x.$$

Both inverse operations are also quasigroups. The operations $(\cdot)$, $(\overset{\ell}{\cdot})$, $(\overset{r}{\cdot})$ and their dual, which are defined by

$$x\overset{s}{\cdot} y:=y\cdot x,\qquad x\overset{s\ell}{\cdot} y:=y\overset{\ell}{\cdot} x,\qquad x\overset{sr}{\cdot} y:=y\overset{r}{\cdot} x, \tag{3}$$

are called *parastrophes* of $(\cdot)$. The defining identities (2) and (3) are called *primary*. An arbitrary σ-parastrophe of $(\cdot)$ can be defined by

$$x_{1\sigma}\overset{\sigma}{\cdot} x_{2\sigma}=x_{3\sigma}:\Leftrightarrow\ x_1\cdot x_2=x_3,$$

where $\sigma\in S_3:=\{\iota,\ell,r,s,s\ell,sr\}$, $\ell:=(13)$, $r:=(23)$, $s:=(12)$. It is easy to verify that

$${}^{\sigma}\Big(\overset{\tau}{\cdot}\Big)=\Big(\overset{\sigma\tau}{\cdot}\Big)$$

holds for all $\sigma,\tau\in S_3$.

1.2. On parastrophic symmetry of an arbitrary proposition.

The relationships (3) imply that each identity of the signature $(\cdot,\ \overset{\ell}{\cdot},\ \overset{r}{\cdot},\ \overset{s}{\cdot},\ \overset{s\ell}{\cdot},\ \overset{sr}{\cdot})$ can be written in the signature $(\cdot,\overset{\ell}{\cdot},\overset{r}{\cdot})$. Nevertheless throughout the article, we consider identities on quasigroups of signature $(\cdot,\ \overset{\ell}{\cdot},\ \overset{r}{\cdot},\ \overset{s}{\cdot},\ \overset{s\ell}{\cdot},\ \overset{sr}{\cdot})$.

The number of different parastrophes of a quasigroup operation $(\cdot)$ depends from its group of parastrophic symmetry $\mathrm{Ps}(\cdot)$. Since $\mathrm{Ps}(\cdot)$ is a subgroup of the symmetric group S_3, then there are six classes of quasigroups. In particular, if $\mathrm{Ps}(\cdot)\supseteq\{\iota,s\}$, then a quasigroup is called *commutative* (*middle symmetric*); if $\mathrm{Ps}(\cdot)\supseteq\{\iota,\ell\}$, then a quasigroup is called *right symmetric*; if $\mathrm{Ps}(\cdot)\supseteq\{\iota,r\}$, then a quasigroup is called *left symmetric*.

Let P be an arbitrary proposition in a class of quasigroups $\mathfrak{A}$. The proposition ${}^{\sigma}P$ is said to be a σ-parastrophe of P, if for every $\tau\in S_3$ it can be obtained from P by replacing $(\overset{\tau}{\cdot})$ with $(\overset{\tau\sigma^{-1}}{\cdot})$. Therefore, ${}^{\sigma}\mathfrak{A}$ denotes the class of all σ-parastrophes of quasigroups from $\mathfrak{A}$ and $\mathrm{Ps}\mathfrak{A}$ denotes its group of parastrophic symmetry.

Theorem 1. [13] *Let $\mathfrak{A}$ be a class of quasigroups, then a proposition P is true in $\mathfrak{A}$ if and only if ${}^{\sigma}P$ is true in ${}^{\sigma}\mathfrak{A}$ for all $\sigma\in S_3$.*

Corollary 1. [13] *Let P be true in a class of quasigroups $\mathfrak{A}$, then ${}^{\sigma}P$ is true in ${}^{\sigma}\mathfrak{A}$ for all $\sigma\in\mathrm{Ps}(\mathfrak{A})$.*

Corollary 2. [13] *Let P be true in a totally symmetric class $\mathfrak{A}$, then ${}^{\sigma}P$ is true in $\mathfrak{A}$ for all $\sigma\in S_3$.*

Corollary 3. [13] *Groups of parastrophic symmetry of parastrophic varieties are conjugate, i.e.,*

$$\mathrm{Ps}({}^{\sigma}\mathfrak{A})=\sigma(\mathrm{Ps}\mathfrak{A})\sigma^{-1}.$$

It is clear that the following proposition holds.

Proposition 1. *If $\mathfrak{A}$ is a variety of commutative quasigroups, then ${}^{\ell}\mathfrak{A}$ and ${}^{r}\mathfrak{A}$ are varieties of left symmetric and right symmetric quasigroups respectively.*

Since S.K. Stein [15] showed that all parastrophes of mediality are primarily equivalent to mediality, then the following theorem is evident.

Theorem 2. *The variety of medial quasigroup is totally symmetric.*

Let $\mathfrak{A}$ be a variety of quasigroups, then ${}^{\sigma}\mathfrak{A}$ is a σ-parastrophe of $\mathfrak{A}$. If $\mathrm{Ps}(\mathfrak{A}) = S_3$, then a variety is called *totally symmetric*; if $\mathrm{Ps}(\mathfrak{A}) = \{\iota, s\}$, i.e., $\mathfrak{A} = {}^{s}\mathfrak{A}$, then $\mathfrak{A}$ is *middle symmetric*; if $\mathrm{Ps}(\mathfrak{A}) = \{\iota, \ell\}$, i.e., $\mathfrak{A} = {}^{\ell}\mathfrak{A}$, then $\mathfrak{A}$ is *right symmetric*; if $\mathrm{Ps}(\mathfrak{A}) = \{\iota, r\}$, i.e., $\mathfrak{A} = {}^{r}\mathfrak{A}$, then $\mathfrak{A}$ is *left symmetric*.

A set of all pairwise parastrophic classes is called a *truss* [14]. A truss of varieties is uniquely defined by one of its varieties. The number of different varieties being parastrophic to $\mathfrak{A}$ is $6/|\mathrm{Ps}\mathfrak{A}|$, that is 1, 2, 3 or 6. A three-element truss is called *middle symmetric*.

Definition 1. *Transformation of the identity $\mathfrak{id}$ to the identity ${}^{\sigma}\mathfrak{id}$ is called a* parastrophic transformation (σ-parastrophic transformation)*, if ${}^{\sigma}\mathfrak{id}$ can be obtained by replacing the main operation in $\mathfrak{id}$ with its σ^{-1}-parastrophe.*

Two identities are called

1. *equivalent*, if they define the same variety;
2. *primarily equivalent*, if one of them can be obtained from the other in a finite number of applications of primary identities (2) – (3) (primary equivalent identities are equivalent);
3. *σ-parastrophic*, if one of them can be obtained from the other by σ-parastrophic transformation;
4. *σ-parastrophically equivalent*, if they define σ-parastrophic varieties;
5. *σ-parastrophically primary equivalent*, if one of them can be obtained in a finite number of applications of primary identities and σ_1-, σ_2-, ..., σ_k- parastrophic transformations such that $\sigma_1\sigma_2\ldots\sigma_k = \sigma$ for some $k \in \mathbb{N}$.

In general case, σ will be omitted. For example, two identities are called *parastrophically equivalent*, if they are σ-parastrophically equivalent for some $\sigma \in S_3$.

1.2. On group isotopes.

A groupoid $(Q;\cdot)$ is called an *isotope of a groupoid* $(Q;+)$ if and only if there exists a triple of bijections (δ, ν, γ), which is called an *isotopism*, such that the relation $x \cdot y := \gamma(\delta^{-1}x + \nu^{-1}y)$ holds. An isotope of a group is called a *group isotope*.

Definition 2. [*11*] *If $(Q;\cdot)$ is a group isotope and 0 is the neutral element of a group $(Q;+)$ and $\alpha 0 = \beta 0 = 0$, then the right part of the formula*

$$x \cdot y = \alpha x + a + \beta y \tag{4}$$

is called a 0-canonical decomposition. We will say that 0 defines the canonical decomposition, $(Q;+)$ is its decomposition group, α and β are its left *and right coefficients, a is its free member.*

Theorem 3. [*11*] *An arbitrary element of a group isotope uniquely defines a canonical decomposition of this isotope.*

Proposition 2. *A triple (α, β, γ) of permutations of a set Q is an autotopism of a commutative group $(Q,+)$ if and only if there exists an automorphism θ of $(Q,+)$ and elements $b, c \in Q$ such that*

$$\alpha = c + \theta x - b, \quad \beta x = b + \theta x, \quad \gamma x = c + \theta x.$$

The following theorem, which is well-known as Toyoda-Bruck theorem, can be found in [1].

Theorem 4. [*1*] *A quasigroup* $(Q;\cdot)$ *satisfies the medial identity if and only if there exists an Abelian group, its automorphisms* α , β *and an element* $a\in Q$ *such that*

$$x\cdot y=\alpha x+a+\beta y,\qquad \alpha\beta=\beta\alpha.$$

It is easy to prove that parastrophes of an isotope of an Abelian group have the following forms

$$\begin{array}{ll} x\overset{\iota}{\cdot}y=\alpha x+a+\beta y; & x\overset{s}{\cdot}y=\beta x+a+\alpha y;\\ x\overset{\ell}{\cdot}y=\alpha^{-1}(x-a-\beta y); & x\overset{s\ell}{\cdot}y=\alpha^{-1}(-\beta x-a+y);\\ x\overset{r}{\cdot}y=\beta^{-1}(-\alpha x-a+y); & x\overset{sr}{\cdot}y=\beta^{-1}(x-a-\alpha y).\end{array}\qquad (5)$$

The following corollary follows from the classification of isotopes of Abelian groups given by H. Krainichuk [3].

Corollary 4. *An isotope* $(Q;\cdot)$ *of an Abelian group with its canonical decomposition* (4) *is left symmetric if and only if* $\beta=-\iota$ *and it is right symmetric if and only if* $\alpha=-\iota$.

Middle symmetry of the truss

In this section, we characterize the truss of varieties defined by the variety of commutative medial quasigroups. This result was announced in [18], [19]. Identities defining commutative medial quasigroups are found by M. Polonijo [8], canonical decomposition of these quasigroups is established by A. Krapež [7].

Proposition 3. [*8*] *The identities*

$$\begin{array}{llll} xy\cdot uv=ux\cdot yv, & (i_1) & xy\cdot uv=uy\cdot vx, & (i_2)\\ xy\cdot uv=xv\cdot uy, & (i_3) & xy\cdot uv=yu\cdot vx, & (i_4)\\ xy\cdot uv=yv\cdot xu, & (i_5) & xy\cdot uv=yv\cdot ux, & (i_6)\\ xy\cdot uv=uy\cdot xv, & (i_7) & xy\cdot uv=vx\cdot yu & (i_8)\end{array}\qquad (6)$$

are equivalent and hold for a quasigroup operation $(\cdot)$ *if and only if the quasigroup is both commutative and medial.*

Theorem 5. [*7*] *Identities* (6) *are equivalent to commutative mediality and define the variety of commutative* T *-quasigroups, i.e.,* $\alpha=\beta$ *in canonical decomposition of these quasigroups.*

Theorem 6. *The truss containing the variety of commutative medial quasigroups is middle symmetric.*

Proof. Let $\mathfrak{C}$ denote the variety being defined by (i_1) from (6). Then according to Proposition 3., $\mathfrak{C}$ is the variety of commutative medial quasigroups. If $(Q;\cdot)$ is an arbitrary quasigroup from $\mathfrak{C}$, then Theorem 5. implies that

$$x\cdot y=\alpha x+a+\alpha y \qquad (7)$$

is a canonical decomposition of $(Q;\cdot)$. Since $(Q;\cdot)$ is commutative, then $(Q;\overset{s}{\cdot})$ coincides with $(Q;\cdot)$. Therefore, $\mathfrak{C}={}^{s}\mathfrak{C}$ and consequently

$${}^{\ell}\mathfrak{C}={}^{\ell}({}^{s}\mathfrak{C})={}^{\ell s}\mathfrak{C}={}^{sr}\mathfrak{C},\qquad {}^{r}\mathfrak{C}={}^{r}({}^{s}\mathfrak{C})={}^{rs}\mathfrak{C}={}^{s\ell}\mathfrak{C}.$$

These equalities imply that the truss of the variety $\mathfrak{C}$ contains not more than three different elements. Furthermore, if two of them coincide, then all elements coincide. Indeed, if ${}^{\ell}\mathfrak{C}=\mathfrak{C}$, then ${}^{r}\mathfrak{C}={}^{s\ell}\mathfrak{C}={}^{s}({}^{\ell}\mathfrak{C})=\mathfrak{C}$. Therefore, it is enough to show that ${}^{\ell}\mathfrak{C}\neq\mathfrak{C}$. For this purpose, we give an example of a quasigroup from $\mathfrak{C}$ such that ℓ-parastrophe of this quasigroup does not belong to $\mathfrak{C}$.

Let us define the operation $(\circ)$ on $\mathbb{Z}_5$, where $\mathbb{Z}_5$ is a ring modulo 5: $x\circ y:=3x+3y$. The equalities (5) imply

$$x\overset{\ell}{\circ}y=3^{-1}(x-3y)=2x-y=2x+4y.$$

Since $2\neq 4$, then by Theorem 5., $(\mathbb{Z}_5;\overset{\ell}{\circ})$ does not belong to $\mathfrak{C}$. Thus, the variety $\mathfrak{C}$ defines three-element truss, that is the truss is middle symmetric. □

Corollary 5. *The varieties* $\mathfrak{C}$, ${}^{\ell}\mathfrak{C}$ *and* ${}^{r}\mathfrak{C}$ *are middle symmetric, left symmetric and right symmetric respectively.*

Proof. According to Theorem 6., $\mathrm{Ps}\mathfrak{C}=\{\iota,s\}$, then Corollary 3. implies

$$\mathrm{Ps}({}^{\ell}\mathfrak{C})=\ell(\mathrm{Ps}\mathfrak{C})\ell=\ell\{\iota,s\}\ell=\{\ell\iota\ell,\ell s\ell\}=\{\iota,r\},$$

$$\mathrm{Ps}({}^{r}\mathfrak{C})=r(\mathrm{Ps}\mathfrak{C})r=r\{\iota,s\}r=\{r\iota r,rsr\}=\{\iota,\ell\}.$$

Thus, this corollary holds. □

Subvarieties of the variety of medial quasigroups

In this section, we find identities defining the variety ${}^{\ell}\mathfrak{C}$ of left symmetric medial quasigroups and the variety ${}^{r}\mathfrak{C}$ of right symmetric medial quasigroups. Also, canonical decompositions of quasigroups of these varieties are given.

Proposition 4. *The identities*

$$\begin{array}{llll} xy\cdot(xu\cdot yv)=uv, & (i_9) & x\cdot(y\cdot(ux\cdot v))=uy\cdot v, & (i_{10})\\ (x\cdot(yu\overset{\ell}{\cdot}v))\cdot u=xy\cdot v, & (i_{11}) & xy\cdot(xu\cdot(y\cdot uv))=v, & (i_{12})\\ xy\cdot(xu\cdot v)=u\cdot yv, & (i_{13}) & x\cdot(yu\cdot(yx\cdot v))=uv, & (i_{14})\\ \big(x\cdot((xy\cdot u)\cdot v)\big)\cdot u=yv, & (i_{15}) & xy\cdot\big(u\cdot(y\cdot(xu\cdot v))\big)=v & (i_{16}) \end{array} \qquad (8)$$

are equivalent and define the variety of left symmetric medial quasigroups.

Proof. The variety $\mathfrak{C}$ is defined by (i_1) from (6), then according to Theorem 1., ${}^{\ell}\mathfrak{C}$ is defined by ℓ-parastrophe of (i_1).

ℓ-parastrophe of (i_1) is

$$(x\overset{\ell}{\cdot}y)\overset{\ell}{\cdot}(u\overset{\ell}{\cdot}v)=(u\overset{\ell}{\cdot}x)\overset{\ell}{\cdot}(y\overset{\ell}{\cdot}v).$$

Replace x with xy and u with uv and use the first identity from (2):

$$x\overset{\ell}{\cdot}u=(uv\overset{\ell}{\cdot}xy)\overset{\ell}{\cdot}(y\overset{\ell}{\cdot}v).$$

Apply the definition of the left division on the right:

$$uv\overset{\ell}{\cdot}xy=(x\overset{\ell}{\cdot}u)\cdot(y\overset{\ell}{\cdot}v).$$

Replace x with xu and y with yv, then by the first identity from (2), we have

$$uv\overset{\ell}{\cdot}(xu\cdot yv)=xy.$$

According to the definition of the left division on the left, we obtain (i_9) from (8), i.e., identities (i_1) and (i_9) are ℓ-parastrophically primary equivalent. Since primary equivalent identities are equivalent, then (i_9) defines the variety ${}^{\ell}\mathfrak{C}$.

In the same way, one can prove that each of the identities (8) is ℓ-parastrophically primary equivalent to the corresponding identity from (6).

Thus, every identity from (8) define the same variety ${}^{\ell}\mathfrak{C}$. That is why these identities are equivalent.

The variety $\mathfrak{C}$ is a subvariety of the variety of commutative quasigroups, then by Proposition 1., ${}^{\ell}\mathfrak{C}$ is a subvariety of the variety of left symmetric quasigroups. The variety of medial quasigroups is totally symmetric (Theorem 2.). Therefore, ${}^{\ell}\mathfrak{C}$ is the variety of left symmetric medial quasigroups as the intersection of varieties of left symmetric and medial quasigroups. □

Theorem 7. *The variety of all left symmetric medial group isotopes is defined by an arbitrary of the identities* (8) *as well as by the following condition*

$$x\cdot y=\alpha x+a-y, \qquad (9)$$

where $(Q;+)$ is an Abelian group, α is its automorphism and $a\in Q$.

Proof. According to Proposition 4., the identities (8) are pairwise equivalent, then it is enough to prove this theorem for one of them.

Let $(Q;\cdot)$ be a quasigroup satisfying the identity (i_9) from (8). By Proposition 4., $(Q;\cdot)$ belongs to ${}^{\ell}\mathfrak{C}$, i.e., this quasigroup is left symmetric and medial. According to Theorem 4., mediality means that $(Q;\cdot)$ is an isotope of an Abelian group $(Q;+)$ and (4) is its canonical decomposition, where α, β are automorphisms of $(Q;+)$. Since $(Q;\cdot)$ is left symmetric, then Corollary 4. implies that $\beta=-\iota$ in its canonical decomposition, i.e., (9) holds.

Visa versa, let $(Q;\cdot)$ be an isotope of an Abelian group $(Q;+)$ and (9) be its canonical decomposition. Then (9) satisfies the identity (i_9). Indeed,

$$xy\cdot(xu\cdot yv)\overset{(9)}{=}\alpha(\alpha x+a-y)+a-(\alpha(\alpha x+a-u)+a-(\alpha y+a-v)).$$

Because the group $(Q;+)$ is Abelian and α is its automorphism, then

$$xy\cdot(xu\cdot yv)=\alpha^2x+\alpha a-\alpha y+a-\alpha^2x-\alpha a+\alpha u-a+\alpha y+a-v=$$
$$=\alpha u+a-\alpha y-a+v=\alpha u+a-(\alpha y+a-v)\overset{(9)}{=}u\cdot yv.$$

Thus, theorem has been proved. □

Proposition 5. *The identities*

$$\begin{array}{llll}(xy\cdot u)\cdot yv=x\cdot uv, & (i_{17}) & ((x\cdot yu)\cdot vu)\cdot y=xv, & (i_{18})\\ x\cdot\big((y\overset{r}{\cdot}(x\cdot uv))\cdot v\big)=yu, & (i_{19}) & ((xy\cdot u)\cdot yv)\cdot uv=x, & (i_{20})\\ (x\cdot yu)\cdot vu=xv\cdot y, & (i_{21}) & ((x\cdot yu)\cdot v)\cdot y=x\cdot vu, & (i_{22})\\ x\cdot((y\overset{r}{\cdot}xu)\cdot v)=y\cdot uv, & (i_{23}) & \big(((x\cdot yu)\cdot v)\cdot y\big)\cdot vu=x. & (i_{24})\end{array}\qquad(10)$$

are equivalent and define the variety of right symmetric medial quasigroups.

Proof. By Theorem 1., the variety ${}^r\mathfrak{C}$ is defined by r-parastrophe of (i_1) from (6).

r-parastrophe of (i_1) is the following identity

$$(x\overset{r}{\cdot}y)\overset{r}{\cdot}(u\overset{r}{\cdot}v)=(u\overset{r}{\cdot}x)\overset{r}{\cdot}(y\overset{r}{\cdot}v).$$

Replace y with xy and v with uv. By the third identity from (2), we have

$$y\overset{r}{\cdot}v=(u\overset{r}{\cdot}x)\overset{r}{\cdot}(xy\overset{r}{\cdot}uv).$$

According to the definition of the right division on the right, we receive

$$xy\overset{r}{\cdot}uv=(u\overset{r}{\cdot}x)\cdot(y\overset{r}{\cdot}v).$$

Replace x with ux and v with yv and apply the third identity from (2):

$$(ux\cdot y)\overset{r}{\cdot}(u\cdot yv)=xv.$$

Using the right division on the left and relabeling variables by the cycle (xyu), we obtain (i_{17}) from (10). It means that (i_1) and (i_{17}) are r-parastrophically primary equivalent. Therefore, (i_{17}) defines the variety ${}^\ell\mathfrak{C}$.

Analogically, one can prove that every identity from (10) defines the variety ${}^r\mathfrak{C}$ and consequently, these identities are equivalent.

The variety of medial quasigroups is totally symmetric (by Theorem 2.) and Proposition 1. implies that ${}^r\mathfrak{C}$ is subvariety of the variety of right symmetric quasigroups. Thus, ${}^r\mathfrak{C}$ is the variety of right symmetric medial quasigroups as the intersection of varieties of right symmetric and medial quasigroups. □

Theorem 8. *The variety of all right symmetric medial group isotopes is defined by an arbitrary of the identities* (10) *as well as by the following condition*

$$x\cdot y=-x+a+\beta y,\qquad(11)$$

where $(Q;+)$ is an Abelian group, β is its automorphism and $a\in Q$.

Proof. According to Proposition 5., the identities (10) are pairwise equivalent, i.e., is enough to prove this theorem for one of them.

Let $(Q;\cdot)$ be a quasigroup satisfying the identity (i_{17}) from (10). By Proposition 5., $(Q;\cdot)$ is a right symmetric medial quasigroup. In virtue of Theorem 4., this quasigroup is an isotope of an Abelian group $(Q;+)$ with its canonical decomposition (4), where α, β are automorphisms of $(Q;+)$. $(Q;\cdot)$ is right symmetric, then Corollary 4. implies that $\alpha=-\iota$ in (4). It means that (11) holds for an arbitrary quasigroup satisfying (i_{17}).

Visa versa, if $(Q;\cdot)$ is an isotope of an Abelian group $(Q;+)$ with its canonical decomposition (11), then (11) satisfies the identity (i_{17}). Indeed,

$$(xy\cdot u)\cdot yv\overset{(11)}{=}-(-(-x+a+\beta y)+a+\beta u)+a+\beta(-y+a+\beta v).$$

Since the group $(Q;+)$ is Abelian and β is its automorphism, then

$$(xy\cdot u)\cdot yv = -x+a+\beta y-a-\beta u+a-\beta y+\beta a+\beta^2 v =$$
$$= -x+a-\beta u+\beta a+\beta^2 v = -x+a+\beta(-u+a+\beta v) \stackrel{(11)}{=} x\cdot uv.$$

Thus, theorem has been proved. □

Results of Theorems 7., 8. were announced by author in [16] and [17].

Conclusions. The truss containing the variety of commutative medial quasigroups is middle symmetric and it contains also varieties of left symmetric and right symmetric medial quasigroups. Identities defining these varieties are given.

The goal of further research is to find all trusses of subvarieties of the variety of medial quasigroups and to describe relationships among them.

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References

[1] *Belousov V.D.* Foundations of quasigroups and loops theory. — Moscow, Nauka, 1967. — 222 p. (in Russian).

[2] *Belyavskaya G.B., Popovich T.V.* Conjugate sets of loops and quasigroups. DC-quasigroups // Bul. Acad. Stiinte Repub. Mold., Mat. — 2012. — No. 1(68). — P. 21–31.

[3] *Krainichuk H.V.* Classification of group isotopes according to their symmetry groups [Electronic resource] // arXiv:1601.07667v1 [math.GR] 28 Jan 2016. — Access mode: http://arxiv.org/pdf/1601.07667v1.pdf.

[4] *Krainichuk H.V.* Existence of skew symmetric varieties of quasigroups // Book of Extended Abstracts. — The International Mathematical Conference on Quasigroups and Loops “Loops’15”, Ohrid, Macedonia, 28 June – 04 July, 2015. — P. 48–49.

[5] *Krainichuk H.V.* Existence of skew symmetric varieties of quasigroups // International Conference of Young Mathematicians. June 3-6, 2015. Abstracts. — K.: Institute of Mathematics of NAS of Ukraine, 2015. — P. 39 (in Ukrainian).

[6] *Krainichuk H. and Tarkovska O.* Semi-symmetric isotopic closure of some group of varieties and the corresponding identities // Bul. Acad. Stiinte Repub. Mold., Mat. (accepted to print).

[7] *Krapež A.* Quadratic level quasigroup equations with four variables I // Publ. Inst. Math., Nouv. Sér. — 2007. — No.81(95). — P. 53–67.

[8] *Polonijo M.* On medial-like identities // Quasigroups and related systems. — 2005. — No.13. — P. 281–288.

[9] *Popovych T.* On conjugate sets of quasigroup // Bul. Acad. Stiinte Repub. Mold., Mat. — 2011. — No. 3(67). — P. 69–76.

[10] *Sade A.* Quasigroupes obéissant à certaines lois // Rev. Fac. Sci. Univ. Istambul. Sér. A. 22. — 1957. — P. 151–184.

[11] *Sokhatsky F.M.* About group isotopes II. // Ukrainian Math. J. — 1995. — No.47(12). — P. 1935–1948.

[12] *Sokhatsky F.M.* // Problem Session 3rd Mile High Conf. on Nonassociative Mathemat., August 16, 2013 [Electronic resource]. — Denver, 2013. — Access mode: http://web.cs.du.edu/ petr/milehigh/2013/problem_session.pdf.

[13] *Sokhatsky F.M.* Symmetry in quasigroup and loop theory [Electronic resource] // 3rd Mile High Conf. on Nonassociative Mathemat., Denver, Colorado, USA, August 11-17, 2013. — Denver, 2013. — Access mode: http://web.cs.du.edu/ petr/milehigh/2013/Sokhatsky.pdf.

[14] *Sokhatsky F.M.* Parastrophic symmetry in quasigroup theory, here.

[15] *Stein S.K.* On the foundation of quasigroups // Trans. Amer. Math. Soc. — 1957. — Vol. 85. — No.1. — P. 228–256.

[16] *Tarkovska O.O.* About one identity of parastrophic mediality // Modern problems of mathematics and its application in natural sciences and information technologies: X intern. conf. for young scientists, April 24-25, 2015. Abstracts of Reports. — Kharkiv: V.N. Karazin Kharkiv National University, 2015. — P. 5 (in Ukrainian).

[17] *Tarkovska O.O.* On canonical decomposition of some parastrophic mediality identities [Electronic resource] // Conference of young scientists “Pidstryhachivski chytannia – 2015”, May 26-28, 2015. — Lviv, 2015. — Access mode: http://iapmm.lviv.ua/chyt2015/theses/Tarkovska.pdf (in Ukrainian).

[18] *Tarkovska O.O.* On parastrophic varieties of parastrophic mediality identities // International Conference of Young Mathematicians. June 3-6, 2015. Abstracts. — K.: Institute of Mathematics of NAS of Ukraine, 2015. — P. 48 (in Ukrainian).

[19] *Tarkovska O.O.* On varieties of parastrophic mediality identities // Book of Extended Abstracts. — The International Mathematical Conference on Quasigroups and Loops "Loops'15", Ohrid, Macedonia, 28 June – 04 July, 2015. — P. 92–93.

[20] *Tarkovska O.O.* Variety of semisymmetry-like medial quasigroups and its subvarieties // Bull. of DonNU, Ser. A. Nat. Sciences. — 2015. — No. 1–2. — P. 78–88.

ПУЧОК, ЯКИЙ МIСТИТЬ МНОГОВИД КОМУТАТИВНИХ МЕДIАЛЬНИХ КВАЗIГРУП

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РЕЗЮМЕ

Розглядається пучок, який мiстить многовид комутативних медiальних квазiгруп. Знайдено тотожностi вiд чотирьох змiнних, що описують многовиди лiво симетричних i право симетричних медiальних квазiгруп. Подано канонiчнi розклади квазiгруп цих многовидiв. Описано парастрофну симетрiю многовидiв.

Ключовi слова: група, квазiгрупа, тотожнiсть, многовид, парастроф, iзотоп, медiальний, комутативний, лiво-симетричний, право-симетричний.

ПУЧОК, СОДЕРЖАЩИЙ МНОГООБРАЗИЕ КОММУТАТИВНЫХ МЕДИАЛЬНЫХ КВАЗИГРУПП

Тарковска Е.А.

РЕЗЮМЕ

Рассматривается пучок, содержащий многообразие коммутативных медиальных квазигрупп. Найдены тождества от четырех переменных, которые описывают многообразия лево симметрических и право симметрических медиальных квазигрупп. Даны канонические разложения квазигрупп этих многообразий. Описано парастрофную симметрию многообразий.

Ключевые слова: группа, квазигруппа, тождество, многообразие, парастроф, изотоп, медиальный, коммутативный, лево симметрический, право симметрический.