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FACTORIZATION OF OPERATIONS OF MEDIAL AND ABELIAN ALGEBRAS

Let A be an $m \times n$ matrix of variables. An n-ary operation f and m-ary operation g are said to *satisfy the medial law* if two results are the same: 1) an application of f to the rows of A then an application of g to the obtained column and 2) an application of g to the columns of A then an application of f to the obtained row. A universal algebra $(A;\Omega)$ is called: *medial* if every two operations from Ω satisfy the medial law; *abelian* if it is medial and has a one-element subalgebra. Criteria for being medial and for being Abelian are found for universal algebras $(A;\Omega)$ which have $0 \in Q$ and $f \in \Omega$ such that the term $f(x_0,\dots,x_n)$ defines a quasigroup operation if all variables are 0 except x_i and x_p and it defines a permutation of Q if all variables are $f(0,\dots,0)$ except x_i or except x_p for some different i, p.

Key words: *mediality, medial law, medial algebras, algebra of endomorphisms, Abelian universal algebra.*

Introduction

A natural appearance of mediality identities and their generalizations are clarified in many works. For instance, in [1, chapter 18], a connection of the identity of generalized mediality with webs and nomograms is shown. A.G. Kurosh [6] proved that endomorphisms of a universal algebra form a universal algebra of the same type if and only if it is Abelian universal algebra, i.e., every two operations of the algebra are connected by the identity of mediality (6) (here, such algebras are called *medial*) and it has a one-element subalgebra.

Let $(Q;+,0)$ be a commutative monoid and let Ω consist of operations being repetition-free composition of $(+)$, some pairwise commuting endomorphisms of the monoid and some elements satisfying (6) $x_{ij} = 0$ for all i,j, then the universal algebra $(Q;\Omega)$ is medial. A counterexample for its reverse statement is not known for the author. A proof of the reverse statement has been given in [3, 1972] for multiary medial quasigroup operations. In [8, 2006], the reverse statement has been proved under assumption that Ω contains an operation which is i- and j-invertible for some i and j. In this article, we generalize this result and prove the reverse statement under the following assumption: a medial algebra $(Q;\Omega)$ has $0 \in Q$ and $f \in \Omega$ such that the term $f(x_0,\dots,x_n)$ defines a quasigroup operation if all variables are 0 except x_i and x_p and it defines a permutation of Q if all variables are $f(0,\dots,0)$ except x_i or except x_p for some different i, p. Other results have been obtained in [4, 2013] and [5, 2014] about the reverse statement under assumption: existence elements with some property of regularity (i.e., invertibility) concerning every operation of a corresponding medial algebra.

Other considerations of medial algebras one can find in the list of references in [4, 5].

1. Preliminary

All operations are supposed to be defined on the same set denoted by Q and called a *carrier*.

An $(n+1)$-ary operation f is called

- i-*invertible*, if the equation $f(a_0,\dots,a_{i-1},x,a_{i+1},\dots,a_n)=a_i$ has a unique solution for all a_0, ..., a_n from Q. In this case, the assignment $(a_0,\dots,a_n)\mapsto x$ defines an operation ${}^{[i]}f$ called i-*th inverse* of f. The i-th invertibility of f means the existence of an operation ${}^{[i]}f$ such that the identities

$$f(x_0,\dots,x_{i-1},{}^{[i]}f(x_0,\dots,x_n),x_{i+1},\dots,x_n)=x_i,$$
$${}^{[i]}f(x_0,\dots,x_{i-1},f(x_0,\dots,x_n),x_{i+1},\dots,x_n)=x_i$$

hold. In other words, the operation f is an i-th invertible element in the i-th symmetric monoid $(\mathcal{O}_{n+1};\underset{i}{\oplus})$ of $(n+1)$-ary operations on Q, where

$$(f\underset{i}{\oplus}h)(x_0,\dots,x_n):=f(x_0,\dots,x_{i-1},h(x_0,\dots,x_n),x_{i+1},\dots,x_n);$$

- *invertible* or a *quasigroup operation*, if it is i-th invertible for all $i=0,\dots,n$;
- *derived* from a group $(Q;\cdot)$, if for all x_0, ..., $x_n\in Q$

$$f(x_0,\dots,x_n)=x_0\cdot x_1\cdot\ldots\cdot x_n.$$

A mapping α of a groupoid $(G;\cdot)$ in a group $(Q;+)$ is called:

- *affine*, if there exists a homomorphism θ of the groupoid $(G;\cdot)$ in the group $(Q;+)$ and elements a, b of $(Q;+)$ such that $\alpha x=a+\theta x+b$ for all $x\in G$;
- *linear*, if there exists an isomorphism θ between $(G;\cdot)$ and $(Q;+)$ and elements $a,b\in Q$ such that $\alpha x=a+\theta x+b$ for all $x\in G$;
- *unitary*, if $\alpha e=0$, where e is a neutral element of the groupoid $(G;\cdot)$.

A sequence $(\alpha_0,\alpha_1,\dots,\alpha_n,\alpha_{n+1})$ of mappings of a set A onto a set B is called a *homotopism* (α_{n+1} is its *principal component*) of an $(n+1)$-ary groupoid $(A;f)$ onto an $(n+1)$-ary groupoid $(B;g)$ if

$$g(\alpha_0x_0,\alpha_1x_1,\dots,\alpha_nx_n)=\alpha_{n+1}f(x_0,x_1,\dots,x_n).$$

If, in addition, the principal component α_{n+1} is: a bijection, then the homotopism is called *cardinal*; an identity transformation (consequently, $A=B$), then it is called *principal.* If all components α_0, α_1, ..., α_n, α_{n+1} are bijections, then the homotopism is an *isotopism.* Respectively, the groupoid $(B;g)$ is called an *homotope* (a *cardinal homotope*, a *principal homotope*, an *isotope*) of $(A;f)$. The relations between the groupoids are called *homotopy*, *cardinal homotopy*, *principal homotopy* and *isotopy* respectively. Note that each cardinal homotopism between binary quasigroups is an isotopism.

It is easy to prove the following statement.

Lemma 1. *A triplet (α,β,γ) of mappings (bijections) of a set Q is a homotopism (isotopism) of a quasigroup $(Q;\cdot)$ onto a group $(Q;+)$ if and only if there exists an endomorphism (automorphism) θ of the group $(Q;+)$ and elements a, $c\in Q$ such that*

$$\alpha x=\theta x+a,\qquad \beta x=-a+\theta x+c,\qquad \gamma x=\theta x+c.$$

Suppose $(n+1)$-ary groupoid $(B; g)$ is a homotope of a quasigroup which is derived from a binary group $(Q; \cdot)$, then $(Q; g)$ will be also called *$(n+1)$-ary homotope of the binary group* $(Q; \cdot)$, briefly a *group homotope*. Therefore, a *cardinal group homotope* is a group homotope such that the principal component is a bijection in the corresponding homotopism.

Let $(Q; f)$ be an arbitrary $(n+1)$-ary groupoid. If there exists a group $(Q; +, 0)$ (not necessary commutative), transformations α_0, α_1, ..., α_n of the carrier and an element a such that

$$f(x_0, x_1, \dots, x_n) = \alpha_0 x_0 + \alpha_1 x_1 + \dots + \alpha_n x_n + a, \tag{1}$$

then $(Q; f)$ is a cardinal homotope of a binary group $(Q; +, 0)$.

The right side of (1) will be called a *0-canonical decomposition* of $(Q; f)$, if $\alpha_0 0 = \ldots = \alpha_n 0 = 0$ and will be denoted by $(0, +, \alpha_0, \dots, \alpha_n, a)$. In this case, α_0, ..., α_n are called *coefficients*, a is a *free member* and $(Q; +)$ is called a *decomposition group*. We will also use the notations

$$\mu_f := \alpha_0 + \alpha_1 + \dots + \alpha_n - \iota, \qquad a_f := a. \tag{2}$$

Lemma 2. *In a cardinal homotope of a binary group, an arbitrary element 0 defines its 0-canonical decomposition.*

Proof. Suppose $(Q; f)$ is an arbitrary $(n+1)$-ary homotope of a binary group $(A; *)$ and let $(\delta_0, \dots, \delta_n, \delta_{n+1})$ be a corresponding homotopism, besides δ_{n+1} is a bijection, i.e.,

$$f(x_0, x_1, \dots, x_n) = \delta_{n+1}^{-1}(\delta_0 x_0 * \delta_1 x_1 * \dots * \delta_n x_n)$$

holds for all x_0, ..., $x_n \in Q$. Define $(\cdot)$ on Q by

$$x \cdot y := \delta_{n+1}^{-1}(\delta_{n+1} x * \delta_{n+1} y), \quad \text{i.e.,} \quad x * y := \delta_{n+1}(\delta_{n+1}^{-1} x \cdot \delta_{n+1}^{-1} y).$$

The first equality means that δ_{n+1} is an isomorphism of the groupoid $(Q; \cdot)$ and the group $(Q; *)$. Therefore, $(Q; \cdot)$ is a group. Using the expression of the operetion $(*)$, the previous relationship can be written as follows

$$f(x_0, x_1, \dots, x_n) = \delta_{n+1}^{-1}\delta_0 x_0 \cdot \delta_{n+1}^{-1}\delta_1 x_1 \cdot \ldots \cdot \delta_{n+1}^{-1}\delta_n x_n.$$

Let 0 be an arbitrary element from Q and 0^{-1} be its inverse in $(Q; \cdot)$. It always exists, since $(Q; \cdot)$ is a group. Define $(+)$ by

$$x + y := x \cdot 0^{-1} \cdot y = \beta^{-1}(\beta x \cdot \beta y), \quad \text{where} \quad \beta x := x \cdot 0^{-1}, \;\; \beta^{-1} x = x \cdot 0.$$

It is easy to verify that $(Q; +)$ is a group and 0 is its neutral element, since $(Q; +)$ is isomorphic to $(Q; \cdot)$. Consequently,

$$f(x_0, x_1, \dots, x_n) = (\delta_{n+1}^{-1}\delta_0 x_0 \cdot 0) + (\delta_{n+1}^{-1}\delta_1 x_1 \cdot 0) + \ldots + (\delta_{n+1}^{-1}\delta_n x_n \cdot 0).$$

For every $i = 0, 1, \dots, n$, we define $a_i := \delta_{n+1}^{-1}\delta_i 0 \cdot 0$ and

$$\gamma_i x_i := (\delta_{n+1}^{-1}\delta_i x_i \cdot 0) - a_i.$$

Therefrom $\gamma_i 0 = 0$ and $\delta_{n+1}^{-1}\delta_i x_i \cdot 0 = \gamma_i x_i + a_i$. Therefore,

$$f(x_0, x_1, \dots, x_n) = \gamma_0 x_0 + a_0 + \gamma_1 x_1 + a_1 + \ldots + \gamma_n x_n + a_n.$$

Introduce some notations:

$$\alpha_0 := \gamma_0, \quad \alpha_1 x := a_0 + \gamma_1 x - a_0, \quad \alpha_2 x := a_0 + a_1 + \gamma_1 x - a_1 - a_0, \ldots$$

$$\alpha_n x := a_0 + \ldots + a_{n-1} + \gamma_1 x - a_{n-1} - \ldots - a_0.$$

It is easy to see that $\alpha_i 0 = 0$ and

$$f(x_0, x_1, \ldots, x_n) = \alpha_0 x_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n + a_0 + a_1 + \ldots + a_n.$$

Substituting $a := a_0 + a_1 + \ldots + a_n$, we obtain (1). Thus, existence of a 0-canonical decomposition has been proved.

A cardinal group homotope will be called *affine*, if all coefficients of its canonical decomposition are endomorphisms of the corresponding decomposition group. If in (1) the free member is 0, i.e. $f(0, \ldots, 0) = 0$, then the homotope will be called *unitary.*

Functional equation of generalized mediality. Universally quantified formula

$$F_1(F_2(x, y), F_3(u, v)) = F_4(F_5(x, u), F_6(y, v)), \tag{3}$$

where F_1, ..., F_6 are function variables, is called a *function equation of generalized mediality.*

This functional equation was solved in [2] on the set of invertible functions defined on an arbitrary set. Namely, the following theorem has been proved.

Theorem 1. *A sextuple* $(f_1, \ldots, f_6)$ *of binary invertible operations defined on a set* Q *is a solution of* (3) *if and only if there exists a commutative group* $(Q;+)$ *and permutations* α, β, γ, δ, λ_2, λ_3, λ_5, λ_6 *of* Q *such that*

$$\left.\begin{array}{ll} f_1(z_1, z_2) = \lambda_2 z_1 + \lambda_3 z_2, & f_4(z_1, z_2) = \lambda_5 z_1 + \lambda_6 z_2, \\ f_2(x, y) = \lambda_2^{-1}(\alpha x + \beta y), & f_5(x, u) = \lambda_5^{-1}(\alpha x + \gamma u), \\ f_3(u, v) = \lambda_3^{-1}(\gamma u + \delta v), & f_6(y, v) = \lambda_6^{-1}(\beta y + \delta v). \end{array}\right\} \tag{4}$$

Corollary 1. *If a sextuple of invertible functions is a solution of the functional equation of general mediality, then all of these functions are isotopic to the same commutative group operation.*

Canonical decompositions of group isotopes. A groupoid $(Q;\cdot)$ is called an *isotope of a groupoid* $(Q';+)$, if there exists an isotopism (δ, ν, γ) such that $x \cdot y := \gamma(\delta^{-1}x + \nu^{-1}y)$ for all $x, y \in Q$. An isotope of a group is called a *group isotope.* A permutation α of a set Q is called *unitary* of a group $(Q;+,0)$, if $\alpha(0) = 0$.

Definition 1. *[5] Let* $(Q;\cdot)$ *be a group isotope and* 0 *be an arbitrary element of* Q. *Then the right side of the formula*

$$x \cdot y = \alpha x + a + \beta y \tag{5}$$

is called a 0-canonical decomposition, *if* $(Q;+)$ *is a group,* 0 *is its neutral element and* $\alpha 0 = \beta 0 = 0$.

In this case, the element 0 is a *defining element*; $(Q;+)$ a *decomposition group*; a its *free member*; α its *left coefficient*; β its *right coefficient*; $J\gamma$, where $\gamma := \alpha\beta^{-1}$, is its *middle coefficient* of the canonical decomposition. Briefly, the canonical decomposition will be denoted by $(+, 0, \alpha, \beta, a)$.

Theorem 2. *[5] An arbitrary element* b *from the carrier uniquely defines* b*-canonical decomposition of an arbitrary group isotope.*

2. Medial and Abelian algebras

One of significant properties of Abelian groups is "homomorphisms of a groupoid onto an Abelian group form an Abelian group". A.G. Kurosh [6] used this property to define a class of universal algebras — Abelian algebras. He proved that this class is a variety whose algebras satisfy a system of identities called *medialities* and have a one-element subalgebra. In [8, 4, 5] a universal algebras with identities of mediality is called a *medial algebra*. Therefore, an *Abelian algebra* is a medial algebra which has a one-element subalgebra.

Identity of mediality. Let

$$\begin{pmatrix} x_{00} & x_{01} & \cdots & x_{0n} \\ x_{10} & x_{11} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m0} & x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

be a matrix of variables. We say that operations f and g of the arity $n+1$ and $m+1$ respectively satisfy the *identity of mediality* if two results are equal for arbitrary meaning of the variables: 1) application f to each row of the matrix and then application g to the obtained results; 2) application g to each column of the matrix and then application f to the obtained results. In other words, the identity

$$\begin{aligned} g(f(x_{00},\dots,x_{0n}),\dots,f(x_{m0},\dots,x_{mn})) = \\ = f(g(x_{00},\dots,x_{m0}),\dots,g(x_{0n},\dots,x_{mn})) \end{aligned} \tag{6}$$

is true.

A universal algebra $(Q;\Omega)$ is called *medial*, if each pair of operations satisfies the identity of mediality. Hence, a groupoid $(Q;f)$ of the arity $n+1$ is *medial*, if it satisfies (6) when $g=f$.

Let an $(n+1)$-ary operation f be defined on Q and let $e\in Q$. The element e will be called:

- *unarily i-invertible* for f, if $\lambda_i(x) := f(\overset{i}{e}, x, \overset{n-i}{e})$ is invertible, i.e., λ_i is a permutation of Q;
- *binarily ip-invertible for* f, if $x\circ y := f(\overset{i}{e}, x, \overset{p-i-1}{e}, y, \overset{n-p}{e})$ is invertible, i.e., $(Q;\circ)$ is a quasigroup.

If a is a unary left invertible in a semigroup $(Q;\cdot)$, i.e. the left translation L_a is a permutation of Q, then $e := L_a^{-1}(a)$ is a left neutral element in $(Q;\cdot)$ and $a^{-1} := L_a^{-2}(a)$ is an inverse to a. The same is true for a right invertible element. Therefore, the 'old' and 'new' notions of invertibility of elements coincide.

An element $0\in Q$ will be called *non-singular* for an operation f defined on Q, if there are integers i and p such that 0 is binarily ip-invertible and the element $a := f(0,\dots,0)$ is unarily i-invertible or unarily p-invertible for f.

Theorem 3. *Let $0\in Q$ be a non-singular element for some operation from Ω. Then the algebra $(Q;\Omega)$ is medial if and only if there exists an Abelian group $(Q;+,0)$, a set E of pairwise commuting endomorphisms of $(Q;+,0)$ and a set $A\subseteq Q$ of elements such that for* *every operation* $g \in \Omega$ *there exist endomorphisms* ψ_0, ψ_1, ..., ψ_m *from* E *and elements* $a_g \in A$ *such that*

$$g(y_0, \dots, y_m) = \psi_0 y_0 + \psi_1 y_1 + \dots + \psi_m y_m + a_g, \tag{7}$$

$$\mu_g(a_h) = \mu_h(a_g) \tag{8}$$

for all g, $h \in \Omega$, *where* $\mu_g := \psi_0 + \psi_1 + \ldots + \psi_m - \iota$.

Proof. Let $0 \in Q$ be a non-singular element for an operation $f \in \Omega$ for some $i < p$. We introduce the following notations: $\overset{s}{c}$ denotes $\underbrace{c, c, \ldots, c}_{s \text{ times}}$ and

$$\left.\begin{array}{ll} \lambda_j x := f(\overset{j}{0}, x, \overset{n-j}{0}),\ j = 0, \dots, n, & a := f(0, \dots, 0), \\ x \circ y := f(\overset{i}{0}, x, \overset{p-i-1}{0}, y, \overset{n-p}{0}), & x * y := f(\overset{i}{a}, x, \overset{p-i-1}{a}, y, \overset{n-p}{a}), \\ \overline{0} := \lambda_p^{-1}(0), \quad \xi x := f(\overset{i}{\overline{0}}, x, \overset{n-i}{\overline{0}}) & \rho_p(x) := f(\overset{p}{a}, x, \overset{n-p}{a}). \end{array}\right\} \tag{9}$$

ip-invertibility of 0 for f implies that $(\circ)$, λ_i, λ_p are invertible operations, that is, $(Q; \circ)$ is a quasigroup and λ_i, λ_p are permutations of the set Q. Non-singularity of 0 for f implies that ρ_p is a permutation of Q.

1°. Consider the following $(n+1) \times (n+1)$ matrix

$$\begin{array}{c} \\ \\ i \\ \\ p \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccc} & & i & & p & & \end{array} \\ \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & x & \dots & y & \dots & 0 \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & u & \dots & v & \dots & 0 \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{pmatrix} \end{array},$$

whose entries are 0 except variables x, y, u, v. Applying f to each row and to each column, we obtain the sequences

$$\overset{i}{a}, x \circ y, \overset{p-i-1}{a}, u \circ v, \overset{n-p}{a} \qquad \text{and} \qquad \overset{i}{a}, x \circ u, \overset{p-i-1}{a}, y \circ v, \overset{n-p}{a}.$$

Since the operation f is medial,

$$f(\overset{i}{a}, x \circ y, \overset{p-i-1}{a}, u \circ v, \overset{n-p}{a}) = f(\overset{i}{a}, x \circ u, \overset{p-i-1}{a}, y \circ v, \overset{n-p}{a}).$$

Taking into account the notations (9), we have an identity:

$$(x \circ y) * (u \circ v) = (x \circ u) * (y \circ v). \tag{10}$$

Putting $x = y = 0$ and $u = v = 0$ in this equality, we have

$$\rho_p(u \circ v) = \lambda_p(u) * \lambda_p(v), \qquad \rho_i(x \circ y) = \lambda_i(x) * \lambda_i(y).$$

According to the condition of the theorem, the element $a := f(0, \dots, 0)$ is i-invertible or p-invertible, i.e., ρ_i or ρ_p is a permutation of Q.

Let ρ_i be a permutation. Since λ_i is a permutation, then second identity implies isotopy of the groupoid $(Q;*)$ and the quasigroup $(Q;\circ)$. Therefore, $(Q;*)$ is also a quasigroup. Putting $v=0$ in the first identity, we obtain $\rho_p\lambda_i=\rho_i\lambda_p$, i.e., $\rho_p=\rho_i\lambda_p\lambda_i^{-1}$. Consequently, ρ_p is a permutation of Q. Analogically, one can prove that $(Q;*)$ is a quasigroup and ρ_i is a permutation if ρ_p is a permutation.

That is why (10) means that the sextuple $(\circ;*;\circ;\circ;*;\circ)$ of invertible operations is a quasigroup solution of the functional equation of generalized mediality. Corollary 1. implies that the quasigroup $(Q;\circ)$ is isotopic to a commutative group. Let (3) be 0-canonical decomposition of $(\circ)$, i.e., $(Q;+,0)$ is a commutative group and $\alpha 0=\beta 0=0$. Note $a=0\circ 0=f(0,\dots,0)$.

Applying f to a variable matrix in which all variables are 0 except entries in p-th column:

$$\begin{pmatrix} 0 & \dots & \overset{p}{x_{0p}} & \dots & 0\\ 0 & \dots & x_{1p} & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & x_{np} & \dots & 0 \end{pmatrix},$$

we obtain

$$\rho_p f(x_{0p},x_{1p},\dots,x_{np})=f(\lambda_p x_{0p},\lambda_p x_{1p},\dots,\lambda_p x_{np}).$$

Using this relationship, we have

$$\rho_p\xi(x)=\rho_p f(\overset{i}{\overline{0}},x,\overset{n-i}{\overline{0}})=f(\overset{i}{0},\lambda_p x,\overset{n-i}{0})=\lambda_i\lambda_p(x),$$

i.e., $\xi=\rho_p^{-1}\lambda_i\lambda_p$. Therefore, ξ is a permutation of Q.

2°. Let r be an arbitrary number from $\overline{1,n}\setminus p$ and let us consider a matrix with the following properties

- i-th column consists of variables but $x_{ri}=0$;
- p-th row consists of $\overline{0}$ but x_{pi} is a variable;
- r-th row consists of 0 but x_{rp} is a variable;
- all other entries are 0.

Consequently, if $r<p$, then the matrix is

$$\begin{matrix} \\ r\\ \\ p\\ \\ \\ \end{matrix}\begin{pmatrix} 0 & \dots & \overset{i}{x_{0i}} & \dots & \overset{p}{0} & \dots & 0\\ 0 & \dots & 0 & \dots & x_{rp} & \dots & 0\\ 0 & \dots & \dots & \dots & \dots & \dots & 0\\ \overline{0} & \dots & x_{pi} & \dots & \overline{0} & \dots & \overline{0}\\ 0 & \dots & \dots & \dots & \dots & \dots & 0\\ 0 & \dots & x_{ni} & \dots & 0 & \dots & 0 \end{pmatrix}.$$

Note, the transformation $\eta x:=f(\overset{r}{0},x,\overset{p-r-1}{0},\overline{0},\overset{n-p}{0})$ are not necessary invertible.

Applying f to r-th row, we obtain $\lambda_p x_{rp}$; applying it to p-th row, ξx_{pi}; applying f to the other rows, we obtain $\lambda_i x_{ji}$, where $j\neq r,p$. Therefore, the left part of (6) is

$$f(\lambda_i x_{0i},\dots,\lambda_i x_{(r-1)i},\lambda_p x_{rp},\lambda_i x_{(r+1)i},\dots,\lambda_i x_{(p-1)i},\xi x_{pi},\lambda_i x_{(p+1)i},\dots,\lambda_i x_{ni}).$$

Applying f to i-th column, we obtain $f(x_{0i},\ldots,x_{(r-1)i},0,x_{(r+1)i},\ldots,x_{ni})$; applying it to p-th column, we obtain ηx_{rp}. Applying f to another columns, we obtain

$$f(\underbrace{0,\ldots,0}_{p\text{ times}},\overline{0},\underbrace{0,\ldots,0}_{n-p\text{ times}})=\lambda_p\overline{0}=\lambda_p\lambda_p^{-1}0=0.$$

Therefore, the medial identity (6) for f of the matrix can be written as follows:

$$\begin{gathered}f(\lambda_i x_{0i},\ \ldots,\lambda_i x_{(r-1)i},\lambda_p x_{rp},\lambda_i x_{(r+1)i},\ldots,\lambda_i x_{(p-1)i},\xi x_{pi},\lambda_i x_{(p+1)i},\ldots,\lambda_i x_{ni})=\\ =f(\overset{i}{0},f(x_{0i},\ldots,x_{(r-1)i},0,x_{(r+1)i},\ldots,x_{ni}),\overset{p-i-1}{0},\eta x_{rp},\overset{n-p}{0}).\end{gathered}\tag{11}$$

According to (9), the right side of the equality can be rewritten as

$$=f(x_{0i},\ldots,x_{(r-1)i},0,x_{(r+1)i},\ldots,x_{ni})\circ\eta x_{rp}$$

Applying (3) to the obtained equality, we get an equivalency of the identity (11) to

$$\begin{gathered}f(\lambda_i x_{0i},\ \ldots,\lambda_i x_{(r-1)i},\lambda_p x_{rp},\lambda_i x_{(r+1)i},\ldots,\lambda_i x_{(p-1)i},\xi x_{pi},\lambda_i x_{(p+1)i},\ldots,\lambda_i x_{ni})=\\ \overset{(3)}{=}\alpha f(x_{0i},\ldots,x_{(r-1)i},0,x_{(r+1)i},\ldots,x_{ni})+a+\beta\eta x_{rp}.\end{gathered}\tag{12}$$

Let $y_j:=\lambda_i x_{ji}$ for all $j\neq r,p$, $y_r:=\lambda_p x_{rp}$, $y_p:=\xi x_{pi}$. Therefrom $x_{ji}=\lambda_i^{-1}y_j$ for all $j\neq r,p$, $x_{rp}=\lambda_p^{-1}y_r$, $x_{pi}=\xi^{-1}y_p$. Hence, if $r<p$, then

$$\begin{gathered}f(y_0,\ldots,y_n)=\alpha f(\lambda_i^{-1}y_0,\ldots,\lambda_i^{-1}y_{r-1},0,\lambda_i^{-1}y_{r+1},\ldots\\ \ldots,\lambda_i^{-1}y_{p-1},\xi^{-1}y_p,\lambda_i^{-1}y_{p+1},\ldots,\lambda_i^{-1}y_n)+a+\beta\eta\lambda_p^{-1}y_r.\end{gathered}\tag{13}$$

Hence, (13) with $y_r=0$ implies

$$\begin{aligned}&f(y_0,\ldots,y_{r-1},0,y_{r+1},\ldots,y_n)-\beta\eta\lambda_p^{-1}0=\\ &=\alpha f(\lambda_i^{-1}y_0,\ldots,\lambda_i^{-1}y_{r-1},0,\lambda_i^{-1}y_{r+1},\ \ldots,\lambda_i^{-1}y_{p-1},\xi^{-1}y_p,\lambda_i^{-1}y_{p+1},\ldots,\\ &\qquad\ldots,\lambda_i^{-1}y_n)+a.\end{aligned}$$

Thus, (13) can be written as

$$f(y_0,\ldots,y_n)=f(y_0,\ldots,y_{r-1},0,y_{r+1},\ldots,y_n)-\beta\eta\lambda_p^{-1}0+\beta\eta\lambda_p^{-1}y_r,$$

Denote $\varphi_r y_r:=-\beta\eta\lambda_p^{-1}0+\beta\eta\lambda_p^{-1}y_r$:

$$f(y_0,\ldots,y_n)=f(y_0,\ldots,y_{r-1},0,y_{r+1},\ldots,y_n)+\varphi_r y_r,\tag{14}$$

Note $\varphi_r 0=0$.

If $r>p$, then the matrix is

$$\begin{array}{c}\\ p\\ \\ r\\ \\ \\ \end{array}\begin{pmatrix} & & i & & p & & \\ 0 & \ldots & x_{0i} & \ldots & 0 & \ldots & 0\\ \overline{0} & \ldots & x_{pi} & \ldots & \overline{0} & \ldots & \overline{0}\\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0\\ 0 & \ldots & 0 & \ldots & x_{rp} & \ldots & 0\\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0\\ 0 & \ldots & x_{ni} & \ldots & 0 & \ldots & 0\end{pmatrix}.$$

Denote $\eta x := f(\overset{p}{0},\overline{0},\overset{r-p-1}{0},x,\overset{n-r}{0})$.

Therefore, the medial identity (6) for f of the matrix can be written as follows:

$$f(\lambda_i x_{0i},\ \ldots,\lambda_i x_{(p-1)i},\xi x_{pi},\lambda_i x_{(p+1)i},\ldots,\lambda_i x_{(r-1)i},\lambda_p x_{rp},\lambda_i x_{(r+1)i},\ldots,\lambda_i x_{ni}) =$$

$$= f(\overset{i}{0},f(x_{0i},\ldots,x_{(r-1)i},0,x_{(r+1)i},\ldots,x_{ni}),\overset{p-i-1}{0},\eta x_{rp},\overset{n-p}{0}).$$

Since we have obtained the same equality as above (see (11)), the relationship (14) is true.

Therefore, we can use it for all $r\neq p$. As a result we obtain

$$f(y_0,\ldots,y_n)=\varphi_0 y_0+\cdots+\varphi_{p-1}y_{p-1}+\varphi_{p+1}y_{p+1}+\cdots+\varphi_n y_n+f(\overset{p}{0},y_p,\overset{n-p}{0}).$$

Moreover,

$$f(\overset{p}{0},y_p,\overset{n-p}{0})=f(\overset{i}{0},0,\overset{p-i-1}{0},y_p,\overset{n-p}{0})\overset{(9)}{=}0\circ y_p\overset{(3)}{=}\alpha 0+a+\beta y_p=\varphi_p y_p+a,$$

where $\varphi_p:=\beta$.

Thus, we have a canonical decomposition for f:

$$f(y_0,\ldots,y_n)=\varphi_0 y_0+\varphi_1 y_1+\cdots+\varphi_n y_n+a. \tag{15}$$

Using this relationship, we get

$$\lambda_i(x)=f(\overset{i}{0},x,\overset{n-i}{0})=\varphi_i x+a.$$

Consequently, $\varphi_i x=\lambda_i(x)-a$ so φ_i is a permutation of Q. The invertibility of φ_p can be verified analogically.

3°. To prove that φ_0, ..., φ_n are endomorphisms of $(Q;+)$, we replace all variables with 0 in (6) ($g=f$) except x_{ri} and x_{rp}, where r is an arbitrary integer from $\{1,\ldots,n\}$:

$$f(\overset{r}{0},f(\overset{i}{0},x_{ri},\overset{p-i-1}{0},x_{rp},\overset{n-p}{0}),\overset{n-r}{0})=f(\overset{i}{0},f(\overset{r}{0},x_{ri},\overset{n-r}{0}),\overset{p-i-1}{0},f(\overset{r}{0},x_{rp},\overset{n-r}{0}),\overset{n-p}{0}).$$

Using (15), we obtain

$$\varphi_r f(\overset{i}{0},x_{ri},\overset{p-i-1}{0},x_{rp},\overset{n-p}{0})+a=\varphi_i f(\overset{r}{0},x_{ri},\overset{n-r}{0})+\varphi_p f(\overset{r}{0},x_{rp},\overset{n-r}{0})+a.$$

Therefrom

$$\varphi_r(\varphi_i x_{ri}+\varphi_p x_{rp}+a)=\varphi_i(\varphi_r x_{ri}+a)+\varphi_p(\varphi_r x_{rp}+a),$$

Since φ_i and φ_p are permutations of the carrier, we replace x_{ri} with $\varphi_i^{-1}x$ and x_{rp} with $\varphi_p^{-1}(y-a)$:

$$\varphi_r(x+y)=\varphi_i R_a\varphi_r\varphi_i^{-1}x+\varphi_p R_a\varphi_r R_{-a}\varphi_p^{-1}y,$$

where $R_a x:=x+a$. It means that $(\varphi_i R_a\varphi_r\varphi_i^{-1},\varphi_p R_a\varphi_r R_{-a}\varphi_p^{-1},\varphi_r)$ is a homotopy of the group $(Q;+,0)$ into itself. According to Lemma 1., there exists an endomorphism θ and element c such that $\varphi_r=R_c\theta$. But $\varphi_r 0=0$, then $\varphi_r=\theta$. Hence φ_r is an endomorphism for all r.

4°. To prove that an arbitrary operation g of the arity $m+1$ has a canonical decomposition on the group $(Q;+)$, we consider a matrix of the size $m\times n$:

$$\begin{array}{c} \\ \\ \\ r \\ \\ \\ \\ \end{array}\begin{pmatrix} 0 & \dots & \overset{i}{x_{0i}} & \dots & \overset{p}{0} & \dots & 0\\ 0 & \dots & \dots & \dots & \dots & \dots & 0\\ 0 & \dots & x_{(r-1)i} & \dots & 0 & \dots & 0\\ 0 & \dots & 0 & \dots & x_{rp} & \dots & 0\\ 0 & \dots & x_{(r+1)i} & \dots & 0 & \dots & 0\\ 0 & \dots & \dots & \dots & \dots & \dots & 0\\ 0 & \dots & x_{mi} & \dots & 0 & \dots & 0 \end{pmatrix}.$$

In this matrix, r is an arbitrary integer from the set $\{1,\dots,m\}$, x_{rp} and x_{ji} with $j\neq r$ are variables, all other entries including x_{ri} are 0. Applying f to the rows and g to the columns, we obtain

$$g(\lambda_i x_{0i},\dots,\lambda_i x_{(r-1)i},\lambda_p x_{rp},\lambda_i x_{(r+1)i},\dots,\lambda_i x_{mi}) =$$
$$= f(\overset{i}{a}, g(x_{0i},\dots,x_{(r-1)i},0,x_{(r+1)i},\dots,x_{mi}), \overset{p-i-1}{a}, \lambda_r x_{rp}, \overset{n-p}{a}).$$

Denote $y_j := \lambda_i x_{ji}$ for $j\neq r$ and $y_r := \lambda_p x_{rp}$ and apply (15):

$$g(y_0,\dots,y_m) = \varphi_i g(\lambda_i^{-1}y_0,\dots,\lambda_i^{-1}y_{r-1},0,\lambda_i^{-1}y_{r+1},\dots,\lambda_i^{-1}y_m) + \chi y_r \qquad (16)$$

for some transformation χ of the carrier Q. In particular, if $y_r=0$ we have

$$\varphi_i g(\lambda_i^{-1}x_{0i},\dots,\lambda_i^{-1}x_{(r-1)i},0,\lambda_i^{-1}x_{(r+1)i},\dots,\lambda_i^{-1}x_{mi}) =$$
$$= g(y_0,\dots,y_{r-1},0,y_{r+1},\dots,y_m) - \chi 0.$$

Putting the obtained relation in (16) and denoting $\psi_r y_r := -\chi 0 + \chi y_r$, the equality (16) can be rewritten as

$$g(y_0,\dots,y_m) = g(y_0,\dots,y_{r-1},0,y_{r+1},\dots,y_m) + \psi_r y_r$$

for some unitary transformation ψ_r, i.e., with the property $\psi_r 0 = 0$. Since r is an arbitrary integer, (7) holds.

5°. To prove that an arbitrary coefficient of an arbitrary operation g is an endomorphism of the group $(Q;+)$, we consider a $m\times n$ matrix

$$\begin{array}{c} \\ \\ i \\ \\ \\ \end{array}\begin{pmatrix} 0 & \dots & \overset{i}{0} & \dots & \overset{p}{0} & \dots & 0\\ 0 & \dots & \dots & \dots & \dots & \dots & 0\\ 0 & \dots & x_{ii} & \dots & x_{ip} & \dots & 0\\ 0 & \dots & \dots & \dots & \dots & \dots & 0\\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{pmatrix}.$$

In the matrix, all entries are 0 except x_{ii} and x_{ip}. Apply f to its rows and g to its columns:

$$g(\overset{r}{a}, f(\overset{i}{0}, x_{ii}, \overset{p-i-1}{0} x_{ip}, \overset{n-p}{0}), \overset{n-r}{a}) = f(\overset{i}{a_g}, g(\overset{i}{0}, x_{ii}, \overset{n-i}{0}), \overset{p-i-1}{a_g}, g(\overset{i}{0}, x_{ip}, \overset{n-i}{0}), \overset{n-p}{a_g}).$$

Taking into account (15) and (7), we obtain

$$\psi_r(\varphi_i x_{ri} + \varphi_p x_{rp} + b) = \gamma x_{ri} + \delta x_{rp}$$

for some permutations γ and δ of Q and an element $b \in Q$. Let $x := \varphi_i x_{ri}$, $y := \varphi_p x_{rp} + b$. Since φ_i and φ_p are permutations of Q,

$$\psi_r(x+y) = \gamma\varphi_i^{-1}(x) + \delta\varphi_p^{-1}L_b^{-1}(y).$$

Therefore, $(\gamma\varphi_i^{-1}, \delta\varphi_p^{-1}L_b^{-1}, \psi_r)$ is an endotopism of the group $(Q;+)$. According to Lemma 1., $\psi_r x = \theta x + c$ for some endomorphism θ and element c. Since $\psi_r 0 = 0$, the element c is 0. Thus, ψ_r is an endomorphism of $(Q;+)$.

6°. Suppose g and h are arbitrary operations from Ω of the arities $m+1$ and $s+1$ respectively. Denote a 0-canonical decomposition of h by

$$h(x_0, \dots, x_s) = \chi_0 x_0 + \chi_1 x_1 + \dots + \chi_s x_s + a_h. \tag{17}$$

To prove that the endomorphisms ψ_j and χ_r commute, consider a matrix whose all entries are 0 except x_{jr}. Applying (7) and canonical decomposition of h, we obtain an equality

$$\psi_j \chi_r(x_{jr}) + b = \chi_r \psi_j(x_{jr}) + c$$

for some elements b and c from Q. The desired property follows from this equality and $b = c$. The last one can be obtained putting $x_{jr} = 0$.

7°. Consider $(m+1)\times(s+1)$ matrix whose all entries are 0. Apply h to its rows and g to its columns:

$$g(a_h, \dots, a_h) = h(a_g, \dots, a_g).$$

Using the canonical decompositions of g and h, we obtain

$$\psi_0 a_h + \psi_1 a_h + \ldots + \psi_m a_h + a_g = \chi_0 a_g + \chi_1 a_g + \ldots + \chi_s a_g + a_h,$$

i.e.,

$$\psi_0 a_h + \psi_1 a_h + \ldots + \psi_m a_h - a_h = \chi_0 a_g + \chi_1 a_g + \ldots + \chi_s a_g - a_g.$$

It means that $\mu_g a_h = \mu_h a_g$.

Thus, we have proved all items of the theorem.

Vice versa, assume all items of the theorem are true. Let us prove the truth of the identities of mediality

$$\begin{aligned} g(h(x_{00}, \dots, x_{0s}), \dots, h(x_{m0}, \dots, x_{ms})) &= \\ = h(g(x_{00}, \dots, x_{m0}), \dots, g(x_{0s}, \dots, x_{ms})) \end{aligned} \tag{18}$$

for every pair (g,h) of operations from Ω. Suppose (7) and (17) are their 0-canonical decompositions, where 0 is the neutral element of the commutative group $(Q;+)$. To prove the identity (6), we calculate its left and right sides:

$$\begin{aligned} & g(h(x_{00}, \dots, x_{0s}), \dots, h(x_{m0}, \dots, x_{ms})) \overset{(7)}{=} \\ & = \psi_0 h(x_{00}, \dots, x_{0s}) + \ldots + \psi_m h(x_{m0}, \dots, x_{ms}) + a_g \overset{(17)}{=} \\ & = \psi_0 \left(\textstyle\sum_{i=0}^s \chi_i x_{0i} + a_h\right) + \ldots + \psi_m \left(\textstyle\sum_{i=0}^s \chi_i x_{mi} + a_h\right) + a_g = \\ & = \left(\textstyle\sum_{i=0}^s \psi_0 \chi_i x_{0i}\right) + \ldots + \left(\textstyle\sum_{i=0}^s \psi_m \chi_i x_{mi}\right) + \psi_0 a_h + \ldots + \psi_m a_h + a_g \overset{(2)}{=} \\ & = \textstyle\sum_{i=0}^m \sum_{j=0}^s \psi_i \chi_j x_{ij} + \mu_g a_h + a_h + a_g. \end{aligned}$$

$$h(g(x_{00},\ldots,x_{m0}),\ldots,g(x_{0s},\ldots,x_{ms})) \overset{(17)}{=}$$
$$= \chi_0 g(x_{00},\ldots,x_{m0}) + \ldots + \chi_m g(x_{0s},\ldots,x_{ms}) + a_h \overset{(7)}{=}$$
$$= \chi_0\left(\textstyle\sum_{i=0}^m \psi_i x_{i0} + a_h\right) + \ldots + \chi_s\left(\textstyle\sum_{i=0}^m \psi_i x_{im} + a_h\right) + a_g =$$
$$= \left(\textstyle\sum_{i=0}^s \chi_0\psi_i x_{i0}\right) + \ldots + \left(\textstyle\sum_{i=0}^s \chi_m\psi_i x_{im}\right) + \chi_0 a_h + \ldots + \chi_m a_h + a_g \overset{(2)}{=}$$
$$= \textstyle\sum_{i=0}^m \sum_{j=0}^s \chi_j\psi_i x_{ij} + \mu_h a_g + a_g + a_h.$$

Since $\chi_j\psi_i = \psi_i\chi_j$ for all i, j and $\mu_h a_g = \mu_g a_h$, the obtained expressions are equal and therefore the identity (18) has been proved.

Corollary 2. *Let $\{0\}$ be a sub-algebra of a universal algebra $(Q;\Omega)$ and the element 0 is binarily invertible for an operation from Ω. Then the algebra $(Q;\Omega)$ is Abelian if and only if there exists an Abelian group $(Q;+,0)$ and a set E of its pairwise commuting endomorphisms such that every operation from Ω is a repetition-free composition of $(+)$ and endomorphisms from E.*

Corollary 3. *Let $(Q;f)$ be an $(n+1)$-ary groupoid which has a binarily invertible element. Then $(Q;f)$ is medial if and only if there exists an Abelian group $(Q;+)$, its pairwise commuting endomorphisms φ_0, ..., φ_n and element $a\in Q$ such that (15) holds.*

Corollary 4. *[3] An $(n+1)$-ary quasigroup $(Q;f)$ is medial if and only if there exists an Abelian group $(Q;+)$, its pairwise commuting automorphisms φ_0, ..., φ_n and an element $a\in Q$ such that (15) holds.*

Corollary 5. *[3] A binary quasigroup $(Q;\circ)$ is medial if and only if there exists an Abelian group $(Q;+)$, its commuting automorphisms φ and ψ and an element $a\in Q$ such that*

$$x\circ y = \varphi x + \psi y + a.$$

Conclusions

In [4], [5] and here, it is proved that every medial algebra having ragular or non-singular element is affine, i.e., there exists a comutative semigroup, a set of its pairwise commuting endomorphisms E and a set of element M such that every operation of the algebra is a repetition-free composition of the semigroup operation, endomorphisms from E and elements from M. The question: *Are there medial algebra with another structure of operations?* is still open.

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РОЗКЛАДИ ОПЕРАЦІЙ МЕДІАЛЬНИХ І АБЕЛЕВИХ АЛГЕБР

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РЕЗЮМЕ

Нехай A — $m\times n$ матриця змінних. Кажуть, що n-арна операція f і m-арна операція g *задовольняють медіальному закону*, якщо таких два результати однакові: 1) застосування f до рядків матриці A, а потім g до отриманого стовпця і 2) застосування g до стовпців матриці A, а потім f до отриманого рядка. Універсальна алгебра $(A;\Omega)$ називається: *медіальною*, якщо кожні дві операції із Ω задовольняють медіальному закону; *абелевою*, якщо вона медіальна і має одноелементну підалгебру. Знайдені критерії 'бути медіальною алгеброю' і критерій 'бути абелевою універсальною алгеброю' для універсальних алгебр $(A;\Omega)$, які мають $0\in Q$ і $f\in\Omega$ такі, що терм $f(x_0,\dots,x_n)$ визначає квазігрупову операцію, коли всі змінні дорівнюють 0 окрім x_i і x_p, а також визначає підстановку, якщо всі змінні є $f(0,\dots,0)$ окрім x_i або окрім x_p для деяких різних i, p.

Ключові слова: *медіальність, медіальний закон, медіальна алгебра, алгебра ендоморфізмів, абелева універсальна алгебра.*

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РАЗЛОЖЕНИЯ ОПЕРАЦИЙ МЕДИАЛЬНЫХ И АБЕЛЕВЫХ АЛГЕБР

РЕЗЮМЕ

Пусть A — $m\times n$ матрица переменных. Говорят, что n-арная операция f и m-арная операция g *удовлетворяют медиальному закону*, если следующих два результаты одинаковы: 1) применение f к строкам матрицы A, а затем g к полученному столпцу и 2) применение g к столпцам матрицы A, а затем f к полученной строке. Универсальная алгебра $(A;\Omega)$ называется: *медиальной*, если каждые две операции из Ω удовлетворяют медиальному закону; *абелевой*, если она медиальна и имеет одноэлементную подалгебру. Найдены критерий 'быть медиальной алгеброй' и критерий 'быть абелевой универсальной алгеброй' для универсальных алгебр $(A;\Omega)$, которые имеют $0\in Q$ и $f\in\Omega$ такие, что терм $f(x_0,\dots,x_n)$ определяет квазигрупповую операцию, если все переменные равны 0 кроме x_i и x_p, а также определяет подстановку, если все переменные равны $f(0,\dots,0)$ кроме x_i или кроме x_p для некоторых разных i, p.

Ключевые слова: *медиальность, медиальный закон, медиальная алгебра, алгебра эндоморфизмов, абелева универсальная алгебра.*