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CLASSIFICATION OF TERNARY QUASIGROUPS ACCORDING TO THEIR PARASTROPHIC SYMMETRY GROUPS, II

The main purpose of this article as well as the previous one [14] is classification of ternary quasigroups according to their parastrophic symmetry groups. Since each of these groups is a subgroup of the symmetric group of degree 4, i.e. S_4, and parastrophic symmetry groups of parastrophic quasigroups are conjugate, then quasigroups whose parastrophic symmetry groups are not pairwise conjugated are considered in these articles. The list of all different parastrophes, sets of identities which define the corresponding varieties, canonical decompositions of ternary group isotopes belonging to these varieties is given.

Key words: *ternary quasigroup, variety of a quasigroup, parastrophic quasigroups, parastrophic symmetry, parastrophic varieties, parastrophic symmetry groups.*

Introduction

One can find all notions and results in [14]. Here we give necessary information for reading only this article.

In [14], it is proved that a class of quasigroups whose parastrophic symmetry groups contain the given subgroup of the group S_4 forms a variety. For example, the trivial subgroup $E := \{\iota\}$ defines the variety of all ternary quasigroups. The quasigroup varieties are parastrophic if and only if the corresponding subgroups are conjugate.

The set $\mathrm{Sub}(S_4)$ of all subgroups of the symmetric group S_4 has 30 elements. The mapping $H \mapsto \tau H \tau^{-1}$ is an action of S_4 on $\mathrm{Sub}(S_4)$. A full set of representatives of the corresponding orbits on $\mathrm{Sub}(S_4)$ is selected. The list contains 11 elements: E, S_2, S_{22}, A_3, $\mathbb{Z}_4$, K_4, C_4, S_3, D_8, A_4, S_4.

In [14], sets of identities defining such varieties are deduced for all subgroups A_3, D_8, S_3, A_4, S_4. The necessary and sufficient conditions for a ternary group isotope to belong to each of these varieties are described. The list of different parastrophes is found for quasigroups whose parastrophic symmetry group is equal to the given subgroup of S_4.

In the current article, these results are obtained for the rest of subgroups: S_2, S_{22}, $\mathbb{Z}_4$, K_4, C_4.

Preliminaries

A ternary operation f is called *invertible* if there exist three operations ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$ called *inverses* or *divisions* such that the identities

$$f({}^{(14)}f(x,y,z),y,z) = x, \quad (1) \qquad {}^{(14)}f(f(x,y,z),y,z) = x, \quad (4)$$

$$f(x,{}^{(24)}f(x,y,z),z) = y, \quad (2) \qquad {}^{(24)}f(x,f(x,y,z),z) = y, \quad (5)$$

$$f(x,y,{}^{(34)}f(x,y,z)) = z, \quad (3) \qquad {}^{(34)}f(x,y,f(x,y,z)) = z \quad (6)$$

hold. In this case, the algebra $(Q; f, {}^{(14)}f, {}^{(24)}f, {}^{(34)}f)$ (in brief, $(Q;f)$) is called a *ternary quasigroup* [2]. It is easy to verify that all divisions of an invertible operation are also invertible and so are their divisions. All of these operations having such connections with the main operation f are called parastrophes of f. Namely, a σ*-parastrophe* of an invertible operation f is called an operation ${}^{\sigma}f$ defined by ${}^{\sigma}f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} \;:\Leftrightarrow\; f(x_1, x_2, x_3) = x_4, \quad \sigma \in S_4,$ where S_4 denotes the group of all bijections of the set $\{0, 1, 2, 3\}$. Therefore in general, every invertible operation has 24 parastrophes. Some of them can coincide. Sometimes it is convenient to use an equivalent form of the formula:

$$ {}^{\sigma}f(x_1, x_2, x_3) = x_4 \;:\Leftrightarrow\; f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}) = x_{4\sigma^{-1}}, \quad \sigma \in S_4. \tag{7} $$

If $4\sigma = 4$, the parastrophe is called principal and can be found by

$$ {}^{\sigma}f(x_1, x_2, x_3) = f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}), \quad \sigma \in S_3. \tag{8} $$

Since for every invertible operation f and for every permutation $\sigma \in S_4$ the relations

$$ {}^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f \quad \text{and} \quad {}^{\iota}f = f \tag{9} $$

hold, then the symmetric group S_4 defines an action on the set Δ_3 of all ternary invertible operations defined on the same carrier. In particular, the fact implies that the number of different parastrophes of an invertible operation is a factor of 24. More precisely, it is equal to $24/|\mathrm{Ps}(f)|$, where $\mathrm{Ps}(f)$ denotes a stabilizer group of f under the action called *parastrophic symmetry group* of the operation f.

Symmetric group S_4. Let S_n denote the symmetric group of degree n, i.e., the group of all bijections from the set $\{1, \dots, n\}$ onto itself. Therefore, S_4 has 24 members:

$$ S_4 = \{\sigma, (04)\sigma, (14)\sigma, (24)\sigma, (34)\sigma \mid \sigma \in S_3\} = \{\sigma, \sigma(04), \sigma(14), \sigma(24), \sigma(34) \mid \sigma \in S_3\}. \tag{10} $$

The equalities (4) and (10) imply that every parastrophe of an invertible operation f is a principal parastrophe of a division of f and equals a division of a principal parastrophe of f.

The set $\mathrm{Sub}(S_4)$ of all subgroups of the symmetric group S_4 has 30 elements. The mapping $H \mapsto \tau H \tau^{-1}$ is an action of S_4 on $\mathrm{Sub}(S_4)$. A full set of representatives of the corresponding orbits on $\mathrm{Sub}(S_4)$ is the following and it contains 11 elements:

$$ S_4, \quad E := \{\iota\}, \quad S_2 := \{\iota, (12)\}, \quad S_{22} := \{\iota, (12)(34)\}, \quad A_3 := \{\iota, (123), (132)\}, $$
$$ \mathbb{Z}_4 := \{\iota, (12)(34), (1423), (1324)\}, \quad K_4 := \{\iota, (12)(34), (13)(24), (14)(23)\}, $$
$$ C_4 := \{\iota, (12), (34), (12)(34)\}, \quad S_3 := \{\iota, (12), (13), (23), (123), (132)\}, $$
$$ D_8 := \{\iota, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}, $$
$$ A_4 := \{\iota, (123), (132), (134)(143), (124), (142), (234), (243), (13)(24), (12)(34), (14)(23)\}. $$

An operation f is called:

- *symmetric*, if some of its parastrophes coincide, i.e. $\mathrm{Ps}(f) \neq \{\iota\}$;
- *asymmetric*, if all its parastrophes are pairwise different, i.e. $\mathrm{Ps}(f) = \{\iota\}$;
- *totally symmetric*, if all its parastrophes coincide, i.e. $\mathrm{Ps}(f) = S_{n+1}$;
- *commutative*, if all its principal parastrophes coincide, i.e. $\mathrm{Ps}(f) \supseteq S_n$;
- *semisymmetric*, if f has at most two different parastrophes, i.e. $\mathrm{Ps}(f) \supseteq A_{n+1}$.
- *dihedrally symmetric* ($n = 3$), if f has at most three different parastrophes, i.e. $\mathrm{Ps}(f) \supseteq D_8$.

1. Group isotopes

A ternary groupoid $(Q;f)$ is called a *group isotope*, if there exists a group $(G;\cdot)$ and bijections α, β, γ, δ from Q to G such that

$$f(x,y,z)=\delta^{-1}(\alpha x\cdot\beta y\cdot\gamma z)$$

for all x, y, z in Q.

Definition 1. *Let $(Q;f)$ be a ternary group isotope and let $(Q;+,0)$ be a group, α_1, α_2, α_3 be its bijections with $\alpha_1 0=\alpha_2 0=\alpha_3 0=0$ and $a\in Q$. If*

$$f(x_1,x_2,x_3)=\alpha_1x_1+\alpha_2x_2+\alpha_3x_3+a, \tag{11}$$

then the tuple $(+,\alpha_1,\alpha_2,\alpha_3,a)$ is called a 0-canonical decomposition *of $(Q;f)$; $(Q;+)$ the* canonical decomposition group*; α_1, α_2, α_3 its* coefficients*; a is called a* free member.

Theorem 1. *[3] An arbitrary element of a ternary group isotope uniquely defines its canonical decomposition.*

Lemma 1. *[3] Let $(Q;f)$ be an arbitrary ternary group isotope and let (1) be its canonical decomposition. Then its divisions and principal parastrophes are*

$$\begin{aligned}
{}^{(14)}f(x_1,x_2,x_3)&=\alpha_1^{-1}(x_1-a-\alpha_3x_3-\alpha_2x_2),\\
{}^{(24)}f(x_1,x_2,x_3)&=\alpha_2^{-1}(-\alpha_1x_1+x_2-a-\alpha_3x_3),\\
{}^{(34)}f(x_1,x_2,x_3)&=\alpha_3^{-1}(-\alpha_2x_2-\alpha_1x_1+x_3-a),\\
{}^{\sigma}f(x_1,x_2,x_3)&=\alpha_1x_{1\sigma^{-1}}+\alpha_2x_{2\sigma^{-1}}+\alpha_{3\sigma^{-1}}x_3+a,\quad \sigma\in S_3.
\end{aligned}\tag{12}$$

A pair $(Q;\Omega)$ is called a *quasigroup algebra*, if Q is a set and Ω is a set of invertible operations defined on Q. A variable is called *quadratic* in an equality, if it has exactly two appearances in it. Let ω be a term, then $\omega(x)$ means that x has an appearance in ω.

Lemma 2. *[4, Theorem 3] Let $(Q;\cdot,\Omega)$ be a quasigroup algebra, additionally, $(Q;\cdot)$ is a group isotope. If the algebra satisfies an identity*

$$\omega_1(x)\cdot\omega_2(y)=\omega_3(y)\cdot\omega_4(x),$$

where the variables x and y are quadratic, then an arbitrary group being isotopic to $(Q;\cdot)$ is commutative.

Lemma 3. *[4, Theorem 3] Let $(Q;+,\Omega)$ be a quasigroup algebra, additionally, $(Q;+,0)$ is a group, $\alpha\in\Omega$ and $\alpha 0=0$. If the algebra satisfies an identity*

$$\alpha(\omega_1(x)+\omega_2(y))=\omega_3(z)+\omega_4(u),$$

then α is either an automorphism of $(Q,+)$, if $z=x$, $u=y$ or anti-automorphism of $(Q,+)$, if $z=y$, $u=x$.

2. Quasigroups with a fixed parastrophic symmetry group

Let $\mathfrak{P}_Q(H)$ and $\mathfrak{Ps}_Q(H)$ denote the sets of all ternary invertible operations defined on a set Q whose parastrophic symmetry group respectively contains the group $H \in S_4$ and equals the group H. Let $\mathfrak{P}(H)$ and $\mathfrak{Ps}(H)$ denote the class of all quasigroups whose parastrophic symmetry group includes the group $H \in S_4$ and equals the group H respectively. It is easy to see that

- a ternary quasigroup $(Q; f)$ belongs to the class $\mathfrak{P}(H)$ if and only if ${}^{\sigma}f = f$ for all σ from a set of generators of the group H therefore, the class of quasigroups $\mathfrak{P}(H)$ is a variety;
- the set $\{\mathfrak{Ps}_Q(H) \mid H \text{ is a subgroup of } S_4\}$ is a partition of the set $\Delta_3(Q)$ of all ternary invertible operations defined on Q.

Lemma 4. *If a non-trivial principal parastrophe of a ternary group isotope f coincides with f, then its canonical decomposition group is commutative.*

3. The group S_{22}.

One of generator sets of the group S_{22} is $\{(12)(34)\}$. Therefore, the variety $\mathfrak{P}(S_{22})$ is defined by the identity

$$ {}^{(12)(34)}f = f. \tag{13} $$

The equality means that for all x_1, x_2, x_3

$$ {}^{(12)(34)}f(x_1, x_2, x_3) = f(x_1, x_2, x_3). $$

Since ${}^{(12)(34)}f = {}^{(12)}({}^{(34)}f)$, then

$$ {}^{(34)}f(x_2, x_1, x_3) = f(x_1, x_2, x_3). \tag{14} $$

Therefrom,

$$ f(x_2, x_1, f(x_1, x_2, x_3)) = x_3. $$

Thus, the following assertion is true.

Proposition 1. *A ternary quasigroup $(Q; f)$ belongs to the variety $\mathfrak{P}(S_{22})$ if and only if*

$$ f(y, x, f(x, y, z)) = z. \tag{15} $$

Proposition 2. *Let $(Q; f)$ be a ternary quasigroup. If $\mathrm{Ps}(f) = S_{22}$, then f, ${}^{(12)}f$, ${}^{(13)}f$, ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(23)}f$, ${}^{(123)}f$, ${}^{(132)}f$, ${}^{(124)}f$, ${}^{(142)}f$, ${}^{(1324)}f$, ${}^{(13)(24)}f$ are all different parastrophes of the operation f.*

Proof. One can verify that

$$ S_4/S_{22} = \{ S_{22}, (12)S_{22}, (13)S_{22}, (14)S_{22}, (24)S_{22}, (23)S_{22}, (123)S_{22}, (132)S_{22}, (124)S_{22}, (142)S_{22}, (1324)S_{22}, (13)(24)S_{22}\} $$

then ι, (12), (13), (14), (24), (23), (123), (132), (124), (142), (1324), $(13)(24)$ are all representatives from S_4/S_{22}. It remains to apply Theorem 3 [14]. □

Theorem 2. *A ternary group isotope* (Q, f) *belongs to* $\mathfrak{P}(S_{22})$ *if and only if there exists a group* $(Q, +, 0)$, *its automorphism* β, *a bijection* α *and an element* $a \in Q$ *such that* $\beta^2 = \iota$, $\alpha 0 = 0$, $-\beta a = a$ *and*

$$f(x_1, x_2, x_3) = \alpha x_1 - \beta\alpha x_2 + \beta x_3 + a. \tag{16}$$

Proof. Let

$$f(x_1, x_2, x_3) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a$$

be a 0-canonical decomposition of f. Using (12), the identity (14) can be written as follows:

$$\alpha_3^{-1}(-\alpha_2 x_1 - \alpha_1 x_2 + x_3 - a) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a. \tag{17}$$

Let $\beta := \alpha_3$ and $\alpha := \alpha_1$, then Lemma 3. implies that β is an automorphism of the group $(Q; +, 0)$ and

$$-\beta^{-1}\alpha_2 x_1 - \beta^{-1}\alpha x_2 + \beta^{-1}x_3 - \beta^{-1}a = \alpha x_1 + \alpha_2 x_2 + \beta x_3 + a.$$

Taking into account the uniqueness of a canonical decomposition, we obtain the identity equivalent to

$$-\beta^{-1}\alpha_2 = \alpha, \qquad -\beta^{-1}\alpha = \alpha_2, \qquad \beta^{-1} = \beta, \qquad -\beta^{-1}a = a.$$

The first and the second equality imply $-\beta\alpha = -\beta^{-1}\alpha$ that is, $\beta = \beta^{-1}$ therefore, $\beta^2 = \iota$. But the third equality means that $\beta^2 = \iota$. Consequently, the identity (17) is equivalent to

$$\beta^2 = \iota, \qquad \alpha_2 = -\beta\alpha, \qquad \beta a = a.$$

□

4. The symmetry group contains $\mathbb{Z}_4$.

At first, we find an identity which describes the variety $\mathfrak{P}(\mathbb{Z}_4)$.

Proposition 3. *A ternary quasigroup* (Q, f) *belongs to* $\mathfrak{P}(\mathbb{Z}_4)$ *if and only if it satisfies the identity*

$$f(x, f(y, z, x), z) = y. \tag{18}$$

Proof. Since (1423) generates the subgroup $\mathbb{Z}_4$ of the symmetric group S_4, then $\mathbb{Z}_4$ is a subgroup of the symmetry group of a ternary quasigroup (Q, f) if and only if the identity

$$^{(1423)}f = f \tag{19}$$

holds in (Q, f). Because $(1423) = (123)(24)$ and $(123)^{-1} = (132)$, then

$$^{(1423)}f = {}^{(123)(24)}f = {}^{(123)}\left({}^{(24)}f\right).$$

Taking into account (2) and $(123)^{-1} = (132)$, we get

$$^{(1423)}f(x_1, x_2, x_3) = {}^{(123)}\left({}^{(24)}f\right)(x_1, x_2, x_3) = {}^{(24)}f(x_3, x_1, x_2).$$

Then the equality (19) can be written as

$$^{(24)}f(x_3, x_1, x_2) = f(x_1, x_2, x_3). \tag{20}$$

Using the definition of (24)-division, we have

$$f(x_3, f(x_1, x_2, x_3), x_2) = x_1.$$

□

Proposition 4. *Let* $(Q;f)$ *be a ternary quasigroup. If* $\mathrm{Ps}(f)=Z_4$, *then* f, ${}^{(12)}f$, ${}^{(13)}f$, ${}^{(23)}f$, ${}^{(14)}f$, ${}^{(24)}f$ *are all different parastrophes of the operation* f.

Proof. Since
$$S_4=Z_4\sqcup(12)Z_4\sqcup(13)Z_4\sqcup(23)Z_4\sqcup(14)Z_4\sqcup(24)Z_4,$$
then ι, (12), (13), (23), (14), (24) are all representatives from S_4/Z_4. □

Theorem 3. *A ternary group isotope* (Q,f) *belongs to* $\mathfrak{P}(\mathbb{Z}_4)$ *if and only if there exists an abelian group* $(Q,+,0)$, *its automorphism* α *and an element* $a\in Q$ *such that* $\alpha^4=\iota$, $\alpha^3a=-a$ *and*
$$f(x_1,x_2,x_3)=\alpha x_1+\alpha^3x_2-\alpha^2x_3+a. \tag{21}$$

Proof. Let (Q,f) be a ternary group isotope and (1) be its 0-decomposition. As it was shown above, the quasigroup (Q,f) belongs to the variety $\mathfrak{P}(\mathbb{Z}_4)$ if and only if (20) holds. Lemma 1. implies that the identity can be written as
$$\alpha_2^{-1}(-\alpha_1x_3+x_1-a-\alpha_3x_2)=\alpha_1x_1+\alpha_2x_2+\alpha_3x_3+a.$$
Fixing x_3 and x_2 one by one in turn and applying Lemma 3., we obtain α_2^{-1} which is both an automorphism and anti-automorphism. Consequently, the group $(Q,+)$ is commutative. Apply α_2 to both sides of the identity:
$$x_1-\alpha_3x_2-\alpha_1x_3-a=\alpha_2\alpha_1x_1+\alpha_2^2x_2+\alpha_2\alpha_3x_3+\alpha_2a.$$
The left and the right sides of the equality are 0-canonical decomposition of the same operation. Therefore according to Theorem 1., the corresponding coefficients and the free members are equal.
$$\iota=\alpha_2\alpha_1,\quad -\alpha_3=\alpha_2^2,\quad -\alpha_1=\alpha_2\alpha_3,\quad -a=\alpha_2a.$$
These equalities are equivalent to the equalities
$$\alpha_2=\alpha_1^{-1},\quad \alpha_3=-\alpha_1^{-2},\quad -\alpha_1=-\alpha_1^{-1}\alpha_1^{-2},\quad \alpha_2a=-a.$$
Let $\alpha:=\alpha_1$. Use the first and the second equalities to form the third one:
$$\alpha_2=\alpha^{-1}=\alpha^3,\qquad \alpha_3=-\alpha^{-2}=-\alpha^2,\qquad \alpha^4=\iota,\qquad \alpha^3a=-a.$$
The theorem has been proved. □

5. The symmetry group contains K_4.

Proposition 5. *A ternary quasigroup* (Q,f) *belongs to* $\mathfrak{P}(K_4)$ *if and only if it satisfies the identities*
$$f(y,x,f(x,y,z))=z,\qquad f(z,f(x,y,z),x)=y. \tag{22}$$

Proof. Since the permutations $(12)(34)$, $(13)(24)$ generate the subgroup K_4 of the symmetric group S_4, then K_4 is a subgroup of the symmetry group of a ternary quasigroup (Q,f) if and only if the equalities
$${}^{(12)(34)}f=f,\qquad {}^{(13)(24)}f=f \tag{23}$$

hold in (Q,f). Applying the formula (2), we get

$$^{(12)(34)}f(x_1,x_2,x_3) = {}^{(34)}f(x_2,x_1,x_3), \qquad ^{((13)(24)}f(x_1,x_2,x_3) = {}^{(24)}f(x_3,x_2,x_1).$$

Therefore, the equalities (23) can be written as identities in the quasigroup (Q,f):

$$^{(34)}f(x_2,x_1,x_3) = f(x_1,x_2,x_3), \qquad ^{(24)}f(x_3,x_2,x_1) = f(x_1,x_2,x_3). \tag{24}$$

Applying the definition of the divisions, we obtain

$$f(x_2,x_1,f(x_1,x_2,x_3)) = x_3, \qquad f(x_3,f(x_1,x_2,x_3),x_1) = x_2.$$

□

Proposition 6. *Let $(Q;f)$ be a ternary quasigroup. If $\mathrm{Ps}(f) = K_4$, then f, $^{(12)}f$, $^{(13)}f$, $^{(14)}f$, $^{(132)}f$, $^{(123)}f$ are all different parastrophes of the operation f.*

Proof. Since

$$S_4 = K_4 \sqcup (12)K_4 \sqcup (13)K_4 \sqcup (14)K_4 \sqcup (132)K_4 \sqcup (123)K_4,$$

then ι, (12), (13), (14), (132), (123) are all representatives from S_4/Z_4. □

Theorem 4. *A ternary group isotope (Q,f) belongs to $\mathfrak{P}(K_4)$ if and only if there exists a group $(Q,+,0)$, its involuting automorphisms α, β and an element $a \in Q$ such that $\alpha a = \beta a = -a$, $\beta\alpha = I_a\alpha\beta$ and*

$$f(x,y,z) = -\beta\alpha x + \alpha y + \beta z + a. \tag{25}$$

Proof. Let (Q,f) be a ternary group isotope and (1) be its 0-decomposition. As it was shown above, the quasigroup (Q,f) belongs to the variety $\mathfrak{P}(\mathbb{Z}_4)$ if and only if (24) holds. Lemma 1. implies that these identities can be written as

$$\alpha_3^{-1}(-\alpha_2 x_1 - \alpha_1 x_2 + x_3 - a) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a,$$

$$\alpha_2^{-1}(-\alpha_1 x_3 + x_2 - a - \alpha_3 x_1) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a.$$

α_3 is an automorphism and α_2 is an anti-automorphism of the group $(Q,+)$ as the result of fixing x_3 and applying Lemma 3. Apply α_3 to the first identity and α_2 to the second one:

$$-\alpha_2 x_1 - \alpha_1 x_2 + x_3 - a = \alpha_3\alpha_1 x_1 + \alpha_3\alpha_2 x_2 + \alpha_3^2 x_3 + \alpha_3 a,$$

$$-\alpha_1 x_3 + x_2 - a - \alpha_3 x_1 = \alpha_2 a + \alpha_2\alpha_3 x_3 + \alpha_2^2 x_2 + \alpha_2\alpha_1 x_1.$$

Denote $I_c := -c + x + c$:

$$-\alpha_2 x_1 - \alpha_1 x_2 + x_3 - a = \alpha_3\alpha_1 x_1 + \alpha_3\alpha_2 x_2 + \alpha_3^2 x_3 + \alpha_3 a,$$

$$-\alpha_1 x_3 + x_2 + I_a(-\alpha_3 x_1) - a = I_{(-\alpha_2 a)}\alpha_2\alpha_3 x_3 + I_{(-\alpha_2 a)}\alpha_2^2 x_2 + I_{(-\alpha_2 a)}\alpha_2\alpha_1 x_1 + \alpha_2 a.$$

The left and the right sides of these equalities are 0-canonical decompositions of the same operations. By Theorem 1. their coefficients and free members are equal:

$$-\alpha_2 = \alpha_3\alpha_1, \qquad -\alpha_1 = \alpha_3\alpha_2, \qquad \iota = \alpha_3^2, \qquad -a = \alpha_3 a,$$

$$-\alpha_1 = I_{(-\alpha_2 a)}\alpha_2\alpha_3, \qquad \iota = I_{(-\alpha_2 a)}\alpha_2^2, \qquad I_a(-\alpha_3) = I_{(-\alpha_2 a)}\alpha_2\alpha_1, \qquad -a = \alpha_2 a.$$

Denote $\alpha := \alpha_2$, $\beta := \alpha_3$:

$$\alpha = -\beta\alpha_1, \qquad \alpha_1 = -\beta\alpha, \qquad \iota = \beta^2, \qquad \beta a = -a,$$

$$\alpha_1 = -I_a\alpha\beta, \qquad \alpha^2 = I_a^{-1}, \qquad -\beta = \alpha\alpha_1, \qquad \alpha a = -a.$$

□

6. The symmetry group contains C_4.

Since $S_4 = C_4 \sqcup (13)C_4 \sqcup (23)C_4 \sqcup (14)C_4 \sqcup (24)C_4 \sqcup (13)(24)C_4$, then all different parastrophes of an invertible operation f with $\mathfrak{Ps}(f) = C_4$ are f, ${}^{(13)}f$, ${}^{(23)}f$, ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(13)(24)}f$.

Proposition 7. *A ternary quasigroup (Q, f) belongs to $\mathfrak{P}(C_4)$ if and only if it satisfies the identities*

$$f(y, x, z) = f(x, y, z), \qquad f(x, y, f(x, y, z)) = z. \tag{26}$$

Proof. Since (12) and (34) generate the subgroup C_4 of the symmetric group S_4, then C_4 is a subgroup of the symmetry group of a ternary quasigroup (Q, f) if and only if the identity

$${}^{(12)}f = f, \qquad {}^{(34)}f = f$$

holds in (Q, f). These equalities could be written as identities in the quasigroup (Q, f):

$${}^{(12)}f(x_1, x_2, x_3) = f(x_1, x_2, x_3), \qquad {}^{(34)}f(x_1, x_2, x_3) = f(x_1, x_2, x_3). \tag{27}$$

Using the definition of parastrophes, we have

$$f(x_2, x_1, x_3) = f(x_1, x_2, x_3), \qquad f(x_1, x_2, f(x_1, x_2, x_3)) = x_3.$$

As a result, we obtain the identities (26). □

Theorem 5. *A ternary group isotope (Q, f) belongs to $\mathfrak{P}(C_4)$ if and only if there exists an abelian group $(Q, +, 0)$, its permutation α and an element $a \in Q$ such that $\alpha 0 = 0$ and*

$$f(x, y, z) = \alpha x + \alpha y - z + a. \tag{28}$$

Proof. Let (Q, f) be a ternary group isotope and (1) be its 0-decomposition. As it was shown above, the quasigroup (Q, f) belongs to the variety $\mathfrak{P}(C_4)$ if and only if (26) holds. Lemma 1. implies that these identities can be written as

$$\alpha_1 x_2 + \alpha_2 x_1 + \alpha_3 x_3 + a = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a,$$

$$\alpha_3^{-1}(-\alpha_2 x_2 - \alpha_1 x_1 + x_3 - a) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a.$$

In obedience to Lemma 2. and Theorem 1., the first identity means that the group $(Q; +)$ is commutative and $\alpha_1 = \alpha_2 := \alpha$. Therefore, in accordance with Lemma 3., the second identity means that $\beta := \alpha_3$ is an automorphism of $(Q; +)$,

$$-\alpha_3^{-1}\alpha = \alpha, \qquad -\alpha_3^{-1}\alpha = \alpha, \qquad \alpha_3^{-1} = \alpha_3, \qquad -\alpha_3^{-1}a = a.$$

i.e., $\alpha_3 = -\iota$. □

7. The symmetry group contains S_2.

Since

$$S_4 = \{\ S_2 \sqcup (13)S_2 \sqcup (23)S_2 \sqcup (14)S_2 \sqcup (24)S_2 \sqcup (34)S_4 \sqcup (134)S_2 \sqcup$$
$$\sqcup (143)S_2 \sqcup (234)S_2 \sqcup (243)S_2 \sqcup (1324)S_4 \sqcup (1423)S_4\},$$

then all different parastrophes of an invertible operation f with $\mathfrak{Ps}(f) = S_2$ are f, ${}^{(13)}f$, ${}^{(23)}f$, ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$, ${}^{(134)}f$, ${}^{(143)}f$, ${}^{(234)}f$, ${}^{(243)}f$, ${}^{(1324)}f$, ${}^{(1423)}f$.

Proposition 8. *A ternary quasigroup* (Q, f) *belongs to* $\mathfrak{P}(S_2)$ *if and only if it satisfies the identity*

$$f(y, x, z) = f(x, y, z). \tag{29}$$

Proof. Evidently. □

Theorem 6. *A ternary group isotope* (Q, f) *belongs to* $\mathfrak{P}(S_2)$ *if and only if there exists a group* $(Q, +, 0)$, *its permutation* α *and an element* $a \in Q$ *such that* $\alpha 0 = 0$ *and*

$$f(x, y, z) = \alpha x + \alpha y + \beta z + a. \tag{30}$$

Proof. Let (Q, f) be a ternary group isotope and (1) be its 0-decomposition. By Theorem 1., the identity means that the group $(Q; +)$ is commutative and $\alpha_1 = \alpha_2 := \alpha$. Let $\beta := \alpha_3$ be an automorphism of $(Q; +)$.

□

8. Conclusion

A class of quasigroups having the same non-trivial parastrophic symmetry group is a variety. There are ten trusses of these varieties. Ten pairwise non-parastrophic varieties are selected for our researh. The respective defining systems of identities in propositions 1, 2, 3, 5, 7 in [14] and propositions 1., 3., 5., 7., 8. in this article are found. In each of these varieties, a group isotope of sub-varieties is described.

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КЛАСИФІКАЦІЯ ТЕРНАРНИХ КВАЗІГРУП ЗА ЇХ ПАРАСТРОФНИМИ ГРУПАМИ СИМЕТРІЙ, II

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РЕЗЮМЕ

Мета цієї та попередньої статті [14] є класифікація тернарних квазігруп за групами їх парастрофної симетрії. Оскільки кожна така група є підгрупою симетричної групи степеня 4, тобто S_4, а парастрофні квазігрупи мають спряжені групи парастрофної симетрії, то в даних працях аналізуються квазігрупи, групи парастрофних симетрій яких попарно неспряжені в S_4. А саме, знаходяться тотожності, які описують клас квазігруп, група парастрофних симетрій яких містить дану підгрупу групи S_4; наводиться список різних парастрофів та знаходяться канонічні розклади операцій групових ізотопів, які містяться в даному многовиді.

Key words: *тернарна квазігрупа, многовид групи, парастрофні квазігрупи, парастрофна симетрія, парастрофні многовиди, парастрофні групи симетрії*

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КЛАССИФИКАЦИЯ ТЕРНАРНЫХ КВАЗИГРУПП В СООТВЕСТВИИ С ИХ ПАРАСТРОФНЫМИ ГРУППАМИ СИММЕТРИЙ, II

РЕЗЮМЕ

Целью этой и предыдущей статей [14] есть классификация тернарных квазигрупп в соответствии с их парастрофными группами симметрий. Так как каждая такая группа является подгруппой симметрической группы симметрий порядка 4, т.е. S_4, а парастрофные квазигруппы имеют сопряженные группы парастрофной симметрии, то в этих работах анализируются квазигруппы, группы парастрофных симметрий которых попарно несопряжены в S_4. Таким образом, найдены тождества, которые описывают класс квазигрупп, группа симметрий которых содержит данную подгруппу группы S_4. Указан список разных парастроф и найдены канонические представления операций групповых изотопов, которые содержатся в данном многовиде.

Key words: *тернарная квазигруппа, многовид группы, парастрофные квазигруппы, парастрофная симметрия, парастрофные многовиды, парастрофные группы симметрий*