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THE BUNCH OF VARIETIES OF INVERSE PROPERTY QUASIGROUPS

In this article, the analysis of the fundamentals of the theory of quasigroups in view of the introduction of the parastrophic symmetry [14] is continued. Parastrophic classes of inverse property quasigroups are considered. The concept of middle IP-quasigroups is introduced. The existence of three different pairwise parastrophic varieties of one-sided-quasigroups: left, right and middle has been proved. Also, it has been proved: if a quasigroup has a j-invertibility property, then its σ-parastrophe has $j\sigma^{-1}$-invertibility property with an invertibility function. An example of a groupoid being a left, right and middle IP-quasigroup simultaneously is given. The group isotopes in each of varieties of the inverse property quasigroups are also described. The trusses containing the varieties of one-sided IP-quasigroups, two-sided IP-quasigroups and three-sided IP-quasigroups are under consideration. These trusses form a bunch.

Key words: *quasigroup, IP-quasigroup, parastrophic symmetry, group isotope, variety, truss, bunch.*

Introduction

The class of inverse property quasigroups has properties similar to groups, at the same time it contains known subclasses such as the class of groups, the class of Moufang loop. In this paper, the analysis of the fundamentals of the theory of quasigroups, namely, inverse property quasigroups is presented. Properties of IP-quasigroups were studied by Belousov V. [1], Bruck R. [7], Pflugfelder H. [6] and Shcherbacov V. [8]. In particular, they considered the elementary relations in the IP-quasigroups.

Here, the class of inverse property quasigroups in view of the concept of parastrophe symmetry introduced in [14] has been analyzed. Namely, all possible parastrophes of the concept of IP-quasigroups are considered, the relations between the corresponding varieties of quasigroups are established. In each of the seven varieties of inverse property quasigroups, the group isotopes are described. In other words, a complete classification of group isotopes for the defining property of IP-quasigroups is given. In this paper the concept of middle IP-quasigroups is introduced, and the isotopes of groups that are left, right, and middle IP-quasigroups are described. The concept of j-invertibility property is introduced. It has been proved: if a quasigroup has a j-invertibility property, then its σ-parastrophe has $j\sigma^{-1}$-invertibility property with some invertibility function. The varieties of inverse property quasigroups with a one-sided inverse property are different and belong to one truss of varieties. It is shown that quasigroups with a two-sided inverse property also belong to one truss and quasigroups with a three-sided inverse property are totally symmetric. These trusses form a bunch. A bunch of the class $\mathfrak{A}$ is said to be a set of all parastrophes of $\mathfrak{A}$ and all their finite intersections [14].

1. Preliminaries

An algebra $(Q;\cdot;\overset{\ell}{\cdot};\overset{r}{\cdot})$ with identities

$$(x\cdot y)\overset{\ell}{\cdot}y=x,\quad (x\overset{\ell}{\cdot}y)\cdot y=x,\quad x\overset{r}{\cdot}(x\cdot y)=y,\quad x\cdot(x\overset{r}{\cdot}y)=y \tag{1}$$

is called a *quasigroup*; the operation $(\cdot)$ is called *main*, $(\overset{\ell}{\cdot})$, $(\overset{r}{\cdot})$ are called *left* and *right divisions*. They are also called *left* and *right inverses* of $(\cdot)$ because they are inverses of $(\cdot)$ in the semigroups $(\mathcal{O}_2, \underset{\ell}{\oplus})$ and $(\mathcal{O}_2, \underset{r}{\oplus})$ respectively, where $\mathcal{O}_2$ denotes the set of all binary operations defined on Q and

$$(f \underset{\ell}{\oplus} g)(x,y) := f(g(x,y),y), \qquad (f \underset{r}{\oplus} g) := f(x, g(x,y)).$$

The set of all invertible binary operations defined on Q is denoted by Δ_2. Each inverse of an invertible operation $(\cdot)$ is also invertible. All such operations are called *parastrophes* of $(\cdot)$ and they are defined by

$$x_{1\sigma} \overset{\sigma}{\cdot} x_{2\sigma} = x_{3\sigma} :\Leftrightarrow x_1 \cdot x_2 = x_3,$$

where $\sigma \in S_3 := \{\iota, \ell, r, s, s\ell, sr\}$, $\ell := (13)$, $r := (23)$, $s := (12)$. In particular, the left and right divisions of $(\cdot)$ are its parastrophes. It is easy to verify that

$$^{\sigma}(\overset{\tau}{\cdot}) = (\overset{\sigma\tau}{\cdot})$$

holds for all $\sigma, \tau \in S_3$, thus S_3 acts on the set Δ_2.

The stabilizer and the orbit of an invertible operation f under this action are called *parastrophic symmetry group* $\mathrm{Ps}(f)$ and truss $\mathrm{Tr}(f)$ respectively. Thus,

$$|\mathrm{Ps}(f)| \cdot |\mathrm{Tr}(f)| = 6.$$

Let P be an arbitrary proposition in a class of quasigroups $\mathfrak{A}$. A proposition $^{\sigma}P$ is said to be a σ*-parastrophe* of P, if it can be obtained from P by replacing the main operation with its σ^{-1}-parastrophe.

Let $^{\sigma}\mathfrak{A}$ denote the class of all σ-parastrophes of quasigroups from $\mathfrak{A}$. A set of all pairwise parastrophic classes is called *a* truss of class $\mathfrak{A}$ [14]

$$Tr(\mathfrak{A}) = \{^{\sigma}\mathfrak{A} \mid \sigma \in S_3\}. \tag{2}$$

A truss of varieties is uniquely defined by one of its varieties.

Theorem 1. *[14] Let $\mathfrak{A}$ be a class of quasigroups, then a proposition P is true in $\mathfrak{A}$ if and only if $^{\sigma}P$ is true in $^{\sigma}\mathfrak{A}$.*

Corollary 1. *[14] An identity $\omega = \upsilon$ defines a variety of quasigroups $\mathfrak{A}$ if and only if σ-parastrophe $^{\sigma}(\omega = \upsilon)$ of this identity defines the variety $^{\sigma}\mathfrak{A}$, where $\sigma \in S_3$.*

A set of all parastrophes of $\mathfrak{A}$ and all their finite intersections is called a *bunch of the class* $\mathfrak{A}$ [14].

A groupoid $(B;\cdot)$ is called an *isotope* of a groupoid $(A;\circ)$, if there are bijections α, β, γ from A to B such that the equality

$$\gamma(x \circ y) = \alpha(x) \cdot \beta(y)$$

holds for all $x, y \in A.$ The triple (α, β, γ) is called an *isotopism* between $(A;\circ)$ and $(B;\cdot)$; bijections α, β, γ are called its *left, right and middle components.*

A quasigroup is called a *group isotope*, if it is isotopic to a group. If there exists a group $(Q;+,0)$ and bijections α, β and also an element a such that $\alpha 0=\beta 0=0$ and

$$x\circ y=\alpha x+a+\beta y \tag{3}$$

for all x, y in Q, then the quaternion $(+,\alpha,\beta,a)$ is called a *0-canonical decomposition* of the group isotope $(Q;\circ)$. In each group isotope, an arbitrary element 0 uniquely defines its 0-canonical decomposition [2].

A quasigroup $(Q;\cdot)$ is called *linear*, if it is a group isotope and coefficients of a canonical decomposition are automorphisms of the canonical decomposition group. The group is also called *linear over the canonical decomposition group.*

Theorem 2. *[9] Each m-order quasigroup being linear over a cyclic group is isomorphic to exactly one quasigroup $(\mathbb{Z}_m;\circ)$, where $\mathbb{Z}_m$ is a ring modulo m, $x\circ y:=ax+by+c$, a, b relatively prime to m, and c is a common factor of m and $a+b-1$.*

Let $(+,\alpha,\beta,a)$ be a canonical decomposition of a group isotope $(Q;\circ)$. Then it is easy to see that all parastrophes of $(Q;\circ)$ have the following forms:

$$\begin{array}{ll} x\overset{\iota}{\cdot}y=\alpha x+a+\beta y, & x\overset{s}{\cdot}y=\beta x+a+\alpha y,\\ x\overset{\ell}{\cdot}y=\alpha^{-1}(x-a-\beta y), & x\overset{s\ell}{\cdot}y=\alpha^{-1}(-\beta x-a+y),\\ x\overset{r}{\cdot}y=\beta^{-1}(-\alpha x-a+y), & x\overset{sr}{\cdot}y=\beta^{-1}(x-a-\alpha y). \end{array} \tag{4}$$

Proposition 1. *[2, Corollary 1] Let $(Q;+,0)$ be a group, $\alpha,\beta_1,\beta_2,\beta_3,\beta_4$ be bijections of Q, besides $\alpha 0=0$ and let*

$$\alpha(\beta_1 x+\beta_2 y)=\beta_3 u+\beta_4 v,$$

where $\{x,y\}=\{u,v\}$ holds for all $x,y\in Q$. Consequently,

- *α is an automorphism of $(Q;+)$, if $u=x$, $v=y$;*
- *α is an anti-automorphism of $(Q;+)$, if $u=y$, $v=x$.*

A transformation α of a set is called *involutive*, if $\alpha^2=\iota$, i.e. $\alpha^{-1}=\alpha$.

Let $(Q;\cdot)$ be a quasigroup. Bijections L_a, R_a, M_a are called *left*, *right* and *middle translations* respectively, if

$$L_a(x)=ax,\qquad R_a(x)=xa,\qquad M_a(x)=y:\Leftrightarrow xy=a. \tag{5}$$

Some of invertible functions have *invertibility properties*, that is they are IP-quasigroups. It is convenient to define these notions as follows.

Definition 1. *A quasigroup $(Q;\cdot)$ is said to have a j*-invertibility property *or $(Q;\cdot)$ is a jIP*-quasigroup *with an* invertibility function ξ, *if the identity*

$$\alpha_1 x_1\cdot\alpha_2 x_2=\alpha_3(x_1\overset{\tau}{\cdot}x_2) \tag{6}$$

is true, where

$$\tau=\begin{cases} r, & if\ j=1,\\ \ell, & if\ j=2,\\ s, & if\ j=3;\end{cases}\qquad\qquad \alpha_k=\begin{cases}\xi, & if\ k=j,\\ \iota, & if\ k\neq j.\end{cases} \tag{7}$$

Thus, $(Q;\cdot)$ is: a *left* IP*-quasigroup* if $j=1$, a *right* IP*-quasigroup* if $j=2$. If $j=3$, the quasigroup will be called a *middle* IP*-quasigroup*. Invertibility functions for the left, right and middle IP-quasigroups are denoted by λ, ρ, μ and will be called *left*, *right* and *middle invertibility functions* respectively. In other words, left, right and middle IP-quasigroups $(Q;\cdot)$ can be defined by

$$x\cdot(\lambda x\cdot y)=y, \quad \text{i.e.,} \quad \lambda x\cdot y = x \overset{r}{\cdot} y; \tag{8}$$

$$(x\cdot\rho y)\cdot y=x, \quad \text{i.e.,} \quad x\cdot\rho y = x \overset{\ell}{\cdot} y; \tag{9}$$

$$x\cdot y=\mu(y\cdot x), \quad \text{i.e.,} \quad \mu(x\cdot y) = x \overset{s}{\cdot} y. \tag{10}$$

Left, right and middle IP-quasigroups are called *one-sided* IP*-quasigroups*.

If some of the functions λ, ρ, μ exist in a quasigroup $(Q;\cdot)$, then they satisfy the following relations. The relations which do not contain μ are well-known (see [1, 6]). That is why, we prove only the relations which contain μ:

1° *Transformations* ρ, λ, μ *are involutive.*

Indeed, $\mu^2(xy)=\mu(\mu(xy))=\mu(yx)=xy$, so $\mu^2=\iota$.

2° $(y\cdot\rho x)\cdot x=y$, $x\cdot(\lambda x\cdot y)=y$, $\rho(x\cdot y)=\lambda y\cdot\lambda x$, $\lambda(x\cdot y)=\rho y\cdot\rho x$;

$\lambda x\cdot\mu(y\cdot x)=y$, $\mu(x\cdot y)\cdot\rho x=y$.

The proof evidently follows from (8), (9), (10).

3° *For an arbitrary* a $L_{\lambda a}=L_a^{-1}$, $R_{\rho a}=R_a^{-1}$, $M_{\mu a}=M_a^{-1}$.

Let $M_{\mu a}(x)=y$ according to the definition $x\cdot y=a$. Apply μ to its sides: $y\cdot x=\mu a$, i.e. $M_{\mu a}(y)=x$, so $y=M_{\mu a}^{-1}(x)$. Thus, $M_{\mu a}^{-1}=M_a$.

4° *For an arbitrary* a we have

$$\begin{array}{llll}
\rho R_a\lambda = L_a^{-1}, & \lambda L_a\rho = R_a^{-1}, & \lambda R_a\rho = L_{\rho a}, & \rho L_a\lambda = R_{\lambda a},\\
\lambda L_a\mu = R_a^{-1}\rho\mu, & \mu R_a\rho = L_a\rho, & \lambda R_a\mu = L_{\rho a}\rho\mu, & \mu R_a\lambda = L_a\lambda,\\
\rho R_a\mu = L_a^{-1}\lambda\mu, & \rho L_a\mu = R_{\lambda a}\lambda\mu, & \mu L_a\lambda = R_a\lambda, & \mu L_a\rho = R_a\rho,\\
L_{\mu a}\rho = M_a, & R_a\lambda = M_{\mu a}^{-1}. & &
\end{array}$$

Indeed, for all $x\in Q$ we obtain

$\lambda L_a\mu(x)=\lambda(a\cdot\mu(x))\overset{2^\circ}{=}\rho\mu(x)\cdot\rho a=R_{\rho a}\rho\mu(x)=R_a^{-1}\rho\mu(x)$;

$\mu R_a\rho(x)=\mu(\rho(x)\cdot a)\overset{(10)}{=}a\cdot\rho(x)=L_a\rho(x)$;

$\lambda R_a\mu(x)=\lambda(\mu(x)\cdot a)\overset{2^\circ}{=}\rho a\cdot\rho\mu(x)=L_{\rho a}\rho\mu(x)$;

$\mu R_a\lambda(x)=\mu(\lambda(x)\cdot a)\overset{(10)}{=}a\cdot\lambda(x)=L_a\lambda(x)$;

$\rho R_a\mu(x)=\rho(\mu(x)\cdot a)\overset{2^\circ}{=}\lambda a\cdot\lambda\mu(x)=L_{\lambda a}\lambda\mu(x)=L_a^{-1}\lambda\mu(x)$;

$\rho L_a\mu(x)=\rho(a\cdot\mu(x))\overset{2^\circ}{=}\lambda\mu(x)\cdot\lambda a=R_{\lambda a}\lambda\mu(x)$;

$\mu L_a\lambda(x)=\mu(a\cdot\lambda(x))\overset{(10)}{=}\lambda(x)\cdot a=R_a\lambda(x)$;

$\mu L_a\rho(x)=\mu(a\cdot\rho(x))\overset{(10)}{=}\rho(x)\cdot a=R_a\rho(x)$;

$L_{\mu a}\rho(x) = \mu a \cdot \rho(x)$. Let $M_a(x) = y$ according to the definition $x \cdot y = a$, we have $L_{\mu a}\rho(x) = \mu(x \cdot y) \cdot \rho(x) \overset{2^\circ}{=} y = M_a(x)$;

$R_a\lambda(x) = \lambda(x) \cdot a$. Let $M_a(x) = y$ according to the definition $x \cdot y = a$, we have $R_a\lambda(x) = \lambda(x) \cdot (x \cdot y) \overset{(8)}{=} y = M_a(x) \overset{3^\circ}{=} M_{\mu a}^{-1}(x)$.

2. Invertibility of group isotopes

Since every group isotope has a canonical decomposition, i.e. the decomposition always exists and is unique, it is natural to find conditions which are equivalent to a one-sided invertibility property.

Theorem 3. *Let $(Q;\circ)$ be a group isotope and (3) be its canonical decomposition, then:*

1) *$(Q;\circ)$ is a right IP-quasigroup with an invertibility function ρ if and only if α is an involutive automorphism of $(Q;+)$ and*

$$\alpha a = -a, \qquad \rho = \beta^{-1} J I_a \alpha\beta. \tag{11}$$

2) *$(Q;\circ)$ is a left IP-quasigroup with an invertibility function λ if and only if β is an involutive automorphism of $(Q;+)$ and*

$$\beta a = -a, \qquad \lambda = \alpha^{-1} J I_a^{-1} \beta\alpha. \tag{12}$$

3) *$(Q;\circ)$ is a middle IP-quasigroup with an invertibility function μ if and only if there exists an anti-automorphism θ such that*

$$\mu x = \theta x + c, \qquad \theta^2 = I_c^{-1}, \qquad \alpha = \theta\beta, \tag{13}$$

where $c := -\theta a + a$.

Proof. 1) Let a group isotope $(Q;\circ)$ be a right IP-quasigroup with invertibility function ρ, e.i., the identity $(y \circ x) \circ \rho x = y$ holds. Using the canonical decomposition of $(Q;\circ)$ (3), we have:

$$\alpha(\alpha y + a + \beta x) + a + \beta\rho x = y, \tag{14}$$

therefore,

$$\alpha(\alpha y + a + \beta x) = y - \beta\rho x - a. \tag{15}$$

Proposition 1 implies that α is an automorphism of $(Q;+,0)$ then

$$\alpha^2 y + \alpha a + \alpha\beta x = y - \beta\rho x - a. \tag{16}$$

When $x = y = 0$, we obtain $\alpha a = -a$ and when $x = 0$ we have $\alpha^2 y = y$, i.e., $\alpha^2 = \iota$.

Substitute the obtained relations in (16):

$$y - a + \alpha\beta x = y - \beta\rho x - a.$$

Reducing y on the left in the equality, we have:

$$-a + \alpha\beta x + a = -\beta\rho x.$$

Wherefrom, $I_a\alpha\beta = J\beta\rho$ and therefore (11) holds.

Conversely, let $(Q;\circ)$ be a group isotope with the canonical decomposition (3) satisfying (11) then

$$\begin{aligned}(y\circ x)\circ\rho x &\overset{(3)}{=} \alpha(\alpha y+a+\beta x)+a+\beta\rho x=\alpha^2 y+\alpha a+\alpha\beta x+a+\beta\rho x=\\ &\overset{(11)}{=} y-a+\alpha\beta x+a+\beta(\beta^{-1}JI_a\alpha\beta)x=y+I_a\alpha\beta x+JI_a\alpha\beta x=\\ &=y+I_a\alpha\beta x-I_a\alpha\beta x=y.\end{aligned}$$

Thus, the group isotope $(Q;\circ)$ has the right inverse property.

2) Let a group isotope $(Q;\circ)$ be a left IP-quasigroup with an invertibility function λ, e.i., the identity $\lambda x\circ(x\circ y)=y$ holds. Using the canonical decomposition of $(Q;\circ)$ (3), we have:

$$\alpha\lambda x+a+\beta(\alpha x+a+\beta y)=y, \tag{17}$$

therefrom

$$\beta(\alpha x+a+\beta y)=-a-\alpha\lambda x+y. \tag{18}$$

Proposition 1 implies that β is an automorphism of $(Q;+,0)$, therefore

$$\beta\alpha x+\beta a+\beta^2 y=-a-\alpha\lambda x+y. \tag{19}$$

If $x=y=0$, we obtain $\beta a=-a$; if $x=0$, we have $\beta^2 y=y$, $\beta^2=\iota$. Substitute the obtained relations in (19):

$$\beta\alpha x-a+y=-a-\alpha\lambda x+y.$$

Canceling y, we get $a+\beta\alpha x-a=-\alpha\lambda x$. Wherefrom, $I_a^{-1}\beta\alpha=J\alpha\lambda$ and thus (12) holds.

Conversely, let $(Q;\circ)$ be a group isotope with the canonical decomposition (3) and the equality (12) holds, then

$$\begin{aligned}\lambda x\circ(x\circ y) &\overset{(3)}{=} \alpha\lambda x+a+\beta(\alpha x+a+\beta y)=\alpha\lambda x+a+\beta\alpha x+\beta a+\beta^2 y=\\ &\overset{(12)}{=} \alpha(\alpha^{-1}JI_a^{-1}\beta\alpha)x+a+\beta\alpha x-a+\beta^2 y=\\ &= -I_a^{-1}\beta\alpha x+I_a^{-1}\beta\alpha x+y=y.\end{aligned}$$

Thus, the group isotope $(Q;\circ)$ has the left inverse property.

3) Let a group isotope $(Q;\circ)$ be a middle IP-quasigroup with an invertibility function μ, i.e., the identity $x\circ y=\mu(y\circ x)$ is true. Using the canonical decomposition of $(Q;\circ)$ (3), we have:

$$\alpha x+a+\beta y=\mu(\alpha y+a+\beta x). \tag{20}$$

Let $c:=\mu 0$ and $\theta:=L_{(-c)}\mu$. We add the element $-c$ from the left to both parts of the equation (20):

$$-c+\alpha x+a+\beta y=L_{(-c)}\mu(\alpha y+a+\beta x)$$

that is

$$R_a L_{(-c)}\alpha(x)+\beta(y)=\theta(R_a\alpha(y)+\beta(x)). \tag{21}$$

Since $\theta(0)=L_{(-c)}\mu 0=-c+\mu 0=-c+c=0$, then by proposition 1. θ is an anti-automorphism of $(Q;+)$. Therefore, $\mu x=\theta x+c$ is an alinear transformation of the group $(Q;+)$ and consequently (20) can be written in the form

$$\alpha x+a+\beta y=\theta\beta x+\theta a+\theta\alpha y+c. \tag{22}$$

In particular, for $x = y = 0$ we get $a = \theta a + c$. For $y = 0$ from the equality (22) we have $\alpha = \theta\beta$. Reducing αx on the left in the equality (22), we have:

$$a + \beta y = \theta a + \theta\alpha y + c \tag{23}$$

because $-a + \theta a = -c$. Then adding $-a$ to both parts of the equality (23), we get:

$$\beta y = -c + \theta\alpha y + c = I_c\theta\alpha y,$$

that is why $\beta = I_c\theta\alpha$. Since $\alpha = \theta\beta$, then $\theta^2 = I_c^{-1}$.

Conversely, let $(Q;\circ)$ be a group isotope with the canonical decomposition (3) and suppose, the conditions of (13) are true. Then

$$\begin{aligned}\mu(y\circ x) = \theta(y\circ x) + c &\overset{(3)}{=} \theta(\alpha y + a + \beta x) + c = \theta\beta x + \theta a + \theta\alpha y + c = \\ &= \alpha x + a - c + \theta\alpha y + c = \alpha x + a + I_c\theta\alpha y = \\ &= \alpha x + a + I_c\theta^2\beta y = \alpha x + a + \beta y = x\circ y.\end{aligned}$$

Thus, $(Q;\circ)$ is a middle IP-quasigroup. □

Applying Theorem 3., we obtain the following assertion.

Corollary 2. *Let $(\mathbb{Z}_m;\circ)$ be a group isotope with the canonical decomposition*

$$x\circ y = ax + c + by, \tag{24}$$

where c is a common factor of m and $a + b - 1$, then:

1) *$(\mathbb{Z}_m;\circ)$ is a right IP-quasigroup with an invertibility function ρ if and only if: $a^2 \equiv 1 \pmod{m}$, $a \equiv -1 \pmod{m/c}$, $\rho(x) = -ax$;*

2) *$(\mathbb{Z}_m;\circ)$ is a left IP-quasigroup with an invertibility function λ if and only if: $b^2 \equiv 1 \pmod{m}$, $b \equiv -1 \pmod{m/c}$, $\lambda(x) = -bx$;*

3) *$(\mathbb{Z}_m;\circ)$ is a middle IP-quasigroup with an invertibility function μ if and only if: $a^2 \equiv b^2 \pmod{m}$, $\mu x = a^{-1}bx - a^{-1}bc + c$.*

Proof. The proof is immediately follows from Theorem 3. and Theorem 2. □

Corollary 3. *If a quasigroup being linear over a finite cyclic group has left and right invertibility properties, then it has a middle invertibility property as well.*

Proof. Let a quasigroup be linear over a finite cyclic group and let it have left and right invertibility properties with functions λ and ρ. This quasigroup is isomorphic to a quasigroup $(Q;\circ)$ which is defined by (24). The items 1) and 2) of Corollary 2. imply $a^2 = b^2$. Therefore according to the item 3), $\mu x = a^{-1}bx - a^{-1}bc + c$ is the middle invertibility function. □

Example 1. Consider the quasigroup $(\mathbb{Z}_8;\circ)$, where $\mathbb{Z}_8$ is a ring modulo 8 and

$$x\circ y := 5x + 4 + 3y.$$

$(\mathbb{Z}_8;\circ)$ is a left and right IP-quasigroup with invertibility functions $\lambda(x) = -3x$ and $\rho(x) = -5x$ respectively.

Let's check if the equalities (8) and (9) are fulfilled:

$$\lambda x \circ (x \circ y) = 5(-3)x + 4 + 3(5x + 4 + 3y) = -15x + 4 + 15x + 12 + y = y,$$

$$(x \circ y) \circ \rho y = 5(5x + 4 + 3y) + 4 + 3(-5)y = x + 4 + 15y + 4 - 15y = x.$$

Thus, $(\mathbb{Z}_8; \circ)$ is a left and right IP-quasigroups with the invertibility functions λ and ρ.

According to Corollary 3. the quasigroup is a middle IP-quasigroup and by Corollary 2. the middle invertibility function μ is

$$\mu x = a^{-1}bx - a^{-1}bc + c = 5^{-1} \cdot 3 \cdot x - 5^{-1} \cdot 3 \cdot 4 + 4 = 7x + 4 + 4 = 7x.$$

Therefore, the quasigroup $(\mathbb{Z}_8; \circ)$ is a left, right and middle IP-quasigroup with the invertibility functions $\lambda(x) = 5x$, $\rho(x) = 3x$, $\mu(x) = 7x$.

Theorem 4. *Let $(Q; \circ)$ be a group isotope and (3) be its canonical decomposition, then:*

1) $(Q; \circ)$ is a left-right IP-quasigroup with a left invertibility function λ and a right invertibility function ρ if and only if the coefficients α and β are involutive automorphisms of $(Q; +)$ respectively, and

$$\alpha a = \beta a = -a, \qquad \lambda = \alpha^{-1} J I_a \beta \alpha, \qquad \rho = \beta^{-1} J I_a \alpha \beta.$$

2) $(Q; \circ)$ is a left-middle IP-quasigroup with a left invertibility function λ and a middle invertibility function μ if and only if β is an involutive anti-automorphism of $(Q; +)$ and there exists an anti-automorphism θ such that

$$\beta a = -a, \qquad \lambda = \alpha^{-1} J I_a^{-1} \beta \alpha,$$

$$\mu x = \theta x + c, \qquad \theta^2 = I_c^{-1}, \qquad \alpha = \theta \beta,$$

where $c := -\theta a + a$.

3) $(Q; \circ)$ is a right-middle IP-quasigroup with a right invertibility function ρ and a middle invertibility function μ if and only if α is an involutive automorphism of $(Q; +)$ and there exists an anti-automorphism θ such that

$$\alpha a = -a, \qquad \rho = \beta^{-1} J I_a \alpha \beta,$$

$$\mu x = \theta x + c, \qquad \theta^2 = I_c^{-1}, \qquad \alpha = \theta \beta,$$

where $c := -\theta a + a$.

Proof. The proof is the same as that of Theorem 3. □

Theorem 5. *Let $(Q; \circ)$ be a group isotope and (3) be its canonical decomposition, then $(Q; \circ)$ is a left-right-middle IP-quasigroup with a left invertibility function λ, a right invertibility function ρ and a middle invertibility function μ if and only if the conditions (11), (12), (13) hold.*

Proof. The proof is follows from proof of Theorem 3. □

3. Varieties of IP-quasigroups

In this subsection, we consider one-sided, two-sided and three-sided IP-quasigroups. First, we prove the following theorem.

Theorem 6. *If a quasigroup has a j-invertibility property, then its σ-parastrophe has a $j\sigma^{-1}$-invertibility property with the same invertibility function.*

Proof. Let a quasigroup $(Q;\cdot)$ has a j-invertibility property that is, there exists a transformation ξ such that (6) is true. Then (6) can be written in the form

$$\alpha_1 x_1 \cdot \alpha_2 x_2 = \alpha_3 x_3 \Leftrightarrow x_1 \overset{\tau}{\cdot} x_2 = x_3.$$

According to the definition of a σ-parastrophe, we have

$$\alpha_{1\sigma} x_{1\sigma} \overset{\sigma}{\cdot} \alpha_{2\sigma} x_{2\sigma} = \alpha_{3\sigma} x_{3\sigma} \Leftrightarrow x_{1\sigma} \overset{\sigma\tau}{\cdot} x_{2\sigma} = x_{3\sigma}.$$

Thus,

$$\alpha_{1\sigma} x_{1\sigma} \overset{\sigma}{\cdot} \alpha_{2\sigma} x_{2\sigma} = \alpha_{3\sigma}(x_{1\sigma} \overset{\sigma\tau}{\cdot} x_{2\sigma}).$$

Replace $\nu := \sigma\tau\sigma^{-1}$, $x := x_{1\sigma}$, $y := x_{2\sigma}$, $(\circ) := (\overset{\sigma}{\cdot})$:

$$\alpha_{1\sigma} x \circ \alpha_{2\sigma} y = \alpha_{3\sigma}(x \overset{\nu}{\circ} y).$$

According to the condition, $\xi = \alpha_j = \alpha_{(j\sigma^{-1})\sigma}$. In the result, we have to make sure that the pair $(j\sigma^{-1}; \sigma\tau\sigma^{-1})$ satisfies the conditions (7). For this purpose, we calculate $(j\sigma^{-1}; \sigma\tau\sigma^{-1})$ for all $j = 1, 2, 3$ and for all $\sigma \in S_3$. All obtained pairs are given in the following table.

$(j,\tau)\setminus\sigma$	ι	s	r	ℓ	$s\ell$	sr
$(1,r)$	$(1,r)$	$(2,\ell)$	$(1,r)$	$(3,s)$	$(3,s)$	$(2,\ell)$
$(2,\ell)$	$(2,\ell)$	$(1,r)$	$(3,s)$	$(2,\ell)$	$(1,r)$	$(3,s)$
$(3,s)$	$(3,s)$	$(3,s)$	$(2,\ell)$	$(1,r)$	$(2,\ell)$	$(1,r)$

The theorem has been proved. □

Corollary 4. *If a quasigroup has i- and j-invertibility properties, then its σ-parastrophe has $i\sigma^{-1}$- $j\sigma^{-1}$-invertibility properties with the same pair of invertibility functions.*

Proof. The proof is follows from Theorem 6. □

If a quasigroup has an i-invertibility property for all $i = 1, 2, 3$, then we say that it has the *total property of invertibility.*

Corollary 5. *If a quasigroup has the total property of invertibility, then each of its parastrophes has this property.*

Proof. The proof is follows from proof of Theorem 6.

Then the table of Theorem 6 can be written in the following form.

ι	*left IP,* λ	*right IP,* ρ	*middle IP,* μ
s-parastrophe	*right IP,* λ	*left IP,* ρ	*middle IP,* μ
ℓ-parastrophe	*middle IP,* λ	*right IP,* ρ	*left IP,* μ
r-parastrophe	*left IP,* λ	*middle IP,* ρ	*right IP,* μ
$s\ell$-parastrophe	*middle IP,* λ	*left IP,* ρ	*right IP,* μ
sr-parastrophe	*right IP,* λ	*middle IP,* ρ	*left IP,* μ

The corollary has been proved. □

Theorem 7. *If a quasigroup, being linear over a finite cyclic group, has two invertibility properties, then it has the third invertibility property too.*

Proof. Since S_3 acts on $\{1,2,3\}$ twice-transitively, then Theorem 6. and the Corollary 3. imply the proof of this theorem. □

4. One-sided IP-quasigroups

Theorem 8. *A truss of varieties of an IP-quasigroup with a one-sided inverse property consists of three varieties: middle, left and right IP-quasigroups. Moreover, if a variety $\mathfrak{A}$ is defined by the identity (10), variety ${}^{\ell}\mathfrak{A}$ is defined by the identity (8) and variety ${}^{r}\mathfrak{A}$ is defined by the identity (9)*

$$Tr(\mathfrak{A}) = \{\mathfrak{A}, {}^{\ell}\mathfrak{A}, {}^{r}\mathfrak{A}\}.$$

$\mathfrak{A} = {}^{s}\mathfrak{A}$	${}^{\ell}\mathfrak{A} = {}^{sr}\mathfrak{A}$	${}^{r}\mathfrak{A} = {}^{s\ell}\mathfrak{A}$
middle IP-quasigroups	*left IP-quasigroups*	*right IP-quasigroups*
$Ps(\mathfrak{A}) = \{\iota, s\}$	$Ps({}^{\ell}\mathfrak{A}) = \{\iota, r\}$	$Ps({}^{r}\mathfrak{A}) = \{\iota, \ell\}$
$x \cdot y = \mu(y \cdot x)$	$\lambda x \cdot (x \cdot y) = y$	$(y \cdot x) \cdot \rho x = y$

Proof. Let $\mathfrak{A}$ be a class of middle IP-quasigroups. We will find an s - parastrophe of this class. According to the definition of a middle IP-quasigroup, we have $x \overset{s}{\circ} y = \mu(y \overset{s}{\circ} x)$, therefore $y \circ x = \mu(x \circ y)$.

Replacing x with y and y with x, we have the equality which defines the variety of middle IP-quasigroups $x \circ y = \mu(y \circ x)$.

Thus, $\mathfrak{A} = {}^{s}\mathfrak{A}$. It implies that $Ps(\mathfrak{A}) = \{\iota, s\} \subseteq S_3$ and the order of the stabilizer is $|Ps(\mathfrak{A})| = 2$.

Therefore, the corresponding truss of varieties $Tr(\mathfrak{A})$ has three elements.

ℓ-parastrophe of class $\mathfrak{A}$: $x \overset{\ell}{\circ} y = \mu(y \overset{\ell}{\circ} x)$. Then we have the identity (8) which defines the variety of left IP-quasigroups ${}^{\ell}\mathfrak{A}$.

r-parastrophe of class $\mathfrak{A}$ is the variety of right IP-quasigroups ${}^{r}\mathfrak{A}$ defined by the identity (9).

Let's find the classes that coincide:

$${}^{sr}\mathfrak{A} = {}^{\ell s}\mathfrak{A} = {}^{\ell}({}^{s}\mathfrak{A}) = {}^{\ell}\mathfrak{A}, \qquad {}^{s\ell}\mathfrak{A} = {}^{rs}\mathfrak{A} = {}^{r}({}^{s}\mathfrak{A}) = {}^{r}\mathfrak{A},$$

$$Ps({}^{\ell}\mathfrak{A}) = \ell(Ps(\mathfrak{A}))\ell = \ell\{\iota, s\}\ell = \{\iota, r\},$$

$$Ps({}^{r}\mathfrak{A}) = r(Ps(\mathfrak{A}))r = r\{\iota, s\}r = \{\iota, \ell\}.$$

In the result, we have the varieties of middle, right and left IP-quasigroups belonging to the truss $Tr(\mathfrak{A}) = \{\mathfrak{A}, {}^{\ell}\mathfrak{A}, {}^{r}\mathfrak{A}\}$. □

Example 2. Consider the quasigroup $(\mathbb{Z}_{15}; \circ)$ with the canonical decomposition

$$x \circ y = 2x + 3 + 4y.$$

$(\mathbb{Z}_{15}; \circ)$ is a left IP-quasigroup with the invertibility function $\lambda = -4$.

Since $\beta a = -a$, then $4 \circ 3 = -3$.

According to Theorem 3., $(\mathbb{Z}_{15};\circ)$ defines the variety of left IP-quasigroups ${}^{\ell}\mathfrak{A}$. Since the variety of left IP-quasigroups is defined by the identity (8), then s-parastrophe of this variety defines the variety of right IP-quasigroups.

Indeed, $\lambda x \overset{s}{\circ} (x\overset{s}{\circ}y) = y$ implies $(y\circ x)\circ\lambda x = y$. From the equalities (4) we get

$$x \overset{s}{\circ} y = 4x + 3 + 2y,$$

Because $2 \circ 3 \neq -3$, then $(\mathbb{Z}_{15};\overset{s}{\circ})$ does not define the variety of left IP-quasigroups ${}^{\ell}\mathfrak{A}$ but defines the variety of right IP-quasigroups ${}^{r}\mathfrak{A}$ by Theorem 8. That is why, ${}^{\ell}\mathfrak{A} \neq {}^{r}\mathfrak{A}$.

ℓ-parastrophe of the left IP-quasigroups implies $\lambda x \overset{\ell}{\circ} (x\overset{\ell}{\circ}y) = y$.

Let $x \overset{\ell}{\circ} y = z$, then $y \circ z = x$. That is why, we have the identity $y \circ z = \lambda(z \circ y)$. Thus, we have the identity which defines the variety of middle IP-quasigroups. The equalities (4) imply

$$x \overset{\ell}{\circ} y = 8x + 6 - 2y.$$

Since $-2 \circ 6 \neq -3$, then $(\mathbb{Z}_{15};\overset{\ell}{\circ})$ does not define the variety of left IP-quasigroups ${}^{\ell}\mathfrak{A}$ but defines the variety of middle IP-quasigroups $\mathfrak{A}$ according to Theorem 8.

That is why ${}^{\ell}\mathfrak{A} \neq \mathfrak{A}$. Consequently, the varieties $\mathfrak{A}, {}^{\ell}\mathfrak{A}, {}^{r}\mathfrak{A}$ are different and belong to the same truss.

Operations $(\circ)$, $(\overset{s}{\circ})$, $(\overset{\ell}{\circ})$ defined on $\mathbb{Z}_{15}$ by the equalities

$$x \circ y := 2x + 3 + 4y, \qquad x \overset{s}{\circ} y := 4x + 3 + 2y, \qquad x \overset{\ell}{\circ} y := 8x + 6 - 2y$$

have left, right and middle inverse properties respectively, in particular $\lambda(x) = 11x$, $\rho(x) = 11x$, $\mu(x) = 11x$.

Indeed, $\lambda x \circ (x \circ y) = 11 \cdot 2x + 3 + 4 \cdot (2x + 3 + 4y) = 22x + 3 + 8x + 12 + 16y = y$,
$(y\circ x)\circ\rho x = 4 \cdot (4y + 3 + 2x) + 3 + 2 \cdot 11x = 16y + 12 + 8x + 3 + 22x = y$,
$\mu(x \circ y) = \mu(8x + 6 - 2y) = -22y + 66 + 88x - 66 + 6 = 8y + 6 - 2x = y \circ x$.

5. Two-sided IP-quasigroups

Theorem 9. *A truss of the varieties of IP-quasigroups with a two-sided inverse property consists of three varieties: left-right, left-middle, right-middle IP-quasigroups.*

Moreover, a variety of left-right IP-quasigroups is defined by identities (8), (9); a variety of left-middle IP-quasigroups is defined by identities (8), (10); a variety of right-middle IP-quasigroups is defined by identities (9), (10).

Proof. We find a set of all pairwise intersections of the classes of the truss $Tr(\mathfrak{A})$. Let $\mathfrak{V} = {}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A}$ be a variety of left-right IP-quasigroups. Let's find all parastrophes of the varieties that coincide:

- a variety of left-right IP-quasigroups

$$\mathfrak{V} = {}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A} = {}^{s}({}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A}) = {}^{s}\mathfrak{V};$$

- a variety of left-middle IP-quasigroups

$${}^{\ell}\mathfrak{V} = {}^{\ell}({}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A}) = {}^{rs}\mathfrak{A} \cap \mathfrak{A} = {}^{r}\mathfrak{A} \cap \mathfrak{A} = {}^{sr}({}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A}) = {}^{sr}\mathfrak{V};$$

- a variety of right-middle IP-quasigroups

$$^{r}\mathfrak{V} = {}^{r}({}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A}) = {}^{\ell}\mathfrak{A} \cap \mathfrak{A} = {}^{s\ell}({}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A}) = {}^{s\ell}\mathfrak{V}.$$

As a result, we have a set of intersections of the varieties of one-sided IP-quasigroups which belongs to the truss of two-sided IP-quasigroups:

$$Tr(\mathfrak{V}) = \{{}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A}, {}^{r}\mathfrak{A} \cap \mathfrak{A}, {}^{\ell}\mathfrak{A} \cap \mathfrak{A}\} = \{\mathfrak{V}, {}^{\ell}\mathfrak{V}, {}^{r}\mathfrak{V}\}.$$

□

For example, two-sided IP-quasigroup is non-commutative loop, if $\mu = \iota$.

Theorem 10. *Three-sided IP-quasigroups are left-right-middle IP-quasigroups. In particular, the variety $\mathfrak{B} = \mathfrak{A} \cap {}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A}$ is a totally symmetric variety.*

Proof. Indeed, easy to verify that

$$^{\sigma}(\mathfrak{A} \cap {}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A}) = \mathfrak{A} \cap {}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A},$$

for all $\sigma \in S_3$. □

Belousov [1] gave an example of a quasigroup with inverse property, where the left and right invertibility functions are different. This quasigroup has a three-sided inverse property.

Let $(G;+)$ be an Abelian group. The operation $(\cdot)$ is defined on the set $G \times G$, putting:

$$(a,b) \cdot (c,d) = (a+c, d-b).$$

The transformations λ, ρ are defined in [1]. The transformation μ is defined as follows:

$$\lambda(a,b) := (-a,-b), \qquad \rho(a,b) := (-a,b), \qquad \mu(a,b) := (a,-b).$$

In [1] it is shown that λ, ρ are left and right invertibility functions respectively and they are different.

We have proved that μ is a middle invertibility function, besides $\mu \neq \lambda$ and $\mu \neq \rho$.

Check the implementation of the identity (10).

$$\mu((a,b)\cdot(c,d)) = \mu(a+c, d-b) = (a+c, -(d-b)) = (a+c, b-d) = (c,d)\cdot(a,b),$$

We take a pair (a,a) and $a \neq -a$. Suppose, $\mu = \lambda$ then $\mu(a,a) = \lambda(a,a)$ in other words $(a,-a) = (-a,-a)$. From this we obtain $a = -a$. This means $\mu \neq \lambda$. That $\mu \neq \rho$ is easily proved.

Thus, μ is a middle invertibility function, moreover it is neither left nor right invertibility function. This example shows that there exist three-sided IP-quasigroups with different invertibility functions.

Theorem 11. *The bunch of the varieties of IP-quasigroups consists of the following varieties:*

1) *The truss of one-sided IP-quasigroups $Tr(\mathfrak{A}) = \{\mathfrak{A}, {}^{\ell}\mathfrak{A}, {}^{r}\mathfrak{A}\}$;*

2) *The truss of two-sided IP-quasigroups $Tr(\mathfrak{V}) = \{\mathfrak{V}, {}^{\ell}\mathfrak{V}, {}^{r}\mathfrak{V}\}$;*

3) *The truss of three-sided IP-quasigroups $\mathfrak{B} = \mathfrak{A} \cap {}^{r}\mathfrak{A} \cap {}^{\ell}\mathfrak{A}$.*

Proof. The proof of item 1) follows from Theorem 8. and Example 2; the proof of item 2) follows from Theorem 9. and the proof of item 3) is the same as that of Theorem 10. □

Conclusion

In this article, the existence of three different pairwise parastrophic varieties of one-sided IP-quasigroups, two-sided IP-quasigroups, three-sided IP-quasigroups has been proved. It has also been proved that these varieties are different. In each of the seven varieties of IP-quasigroups, the group isotopes are described and a complete classification of group isotopes for the defining property of IP-quasigroups is given. The examples of a groupoid being a left, right and middle IP-quasigroup simultaneously and grupoid with one-sided inverse property are given. The trusses containing the varieties of one-sided IP-quasigroups, two-sided IP-quasigroups and three-sided IP-quasigroups are under consideration. It has been proved that the bunch of the varieties of inverse property quasigroups consists of seven varieties.

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В'ЯЗКА МНОГОВИДІВ КВАЗІГРУП З ВЛАСТИВІСТЮ ОБОРОТНОСТІ

РЕЗЮМЕ

В даній статті ми продовжуємо аналізувати основи теорії квазігруп з огляду на введене в [14] поняття парастрофної симетрії. Розглядаються парастрофні класи квазігруп з властивістю оборотності. Введено поняття середньої IP-квазігрупи. Доведено існування трьох різних попарно-парастрофних многовиди односторонніх IP-квазігруп: лівих, правих і середніх. Також доведено, що якщо квазігрупа має j властивість оборотності, то її σ-парастроф має $j\sigma^{-1}$ властивість оборотності з деякою функцією оборотності. Наведено приклад групоїда, що одночасно є лівою, правою і середньою IP-квазігрупою. Також у кожному з многовидів квазігруп ми описали групові ізотопи. Розглядаються пучки, що містять многовиди односторонніх IP-квазігруп, двосторонніх IP-квазігруп і тристоронніх IP-квазігруп. Ці пучки утворюють в'язку.

Ключові слова: *квазігрупа, IP-квазігрупа, парастрофна симетрія, груповий ізотоп, многовид, пучок, в'язка.*

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СВЯЗКА МНОГООБРАЗИЙ КВАЗИГРУПП СО СВОЙСТВОМ ОБРАТИМОСТИ

РЕЗЮМЕ
В данной статье мы продолжаем анализ основ теории квазигрупп учитывая введенное в [14] понятие парастрофной симметрии. Рассматриваются парастрофные классы квазигрупп со свойством обратимости. Введено понятие средней IP-квазигруппы. Существование трех разных попарно-парастрофных многообразий односторонних IP-квазигрупп: левых, правых и средних доказано. Также доказано, что если квазигруппа имеет j свойство обратимости, то ее σ-парастроф имеет $j\sigma^{-1}$ свойство обратимости с некоторой функцией обратимости. Приведен пример группоида, который одновременно является левой, правой и средней IP-квазигруппой. Также описаны групповые изотопы в каждом из многообразий IP-квазигрупп. Рассматриваются пучки, содержащие многообразия односторонних IP-квазигрупп, двусторонних IP-квазигрупп и трехсторонних IP-квазигрупп. Эти пучки образуют связку.

Ключевые слова: *квазигруппа, IP-квазигруппа, парастрофная симметрия, групповой изотоп, многообразие, пучок, связка.*