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CENTRAL CHARGE AND TOPOLOGICAL INVARIANT OF CALABI-YAU MANIFOLDS

F-theory, as a 12-dimensional theory that is a contender of the Theory of Everything, should be compactified into elliptically fibered threefolds or fourfolds of Calabi-Yau. Such manifolds have an elliptic curve as a fiber, and their bases may have singularities. We considered orbifold as simplest non-flat construction. Blow up modes of orbifold singularities can be considered as coordinates of complexified Kahler moduli space. Quiver diagrams are used for describing D-branes near orbifold point. In this case it is possible to calculate Euler character defined through $\text{Ext}^i(A, B)$ groups and coherent sheaves A, B over projective space, which are representations of orbifold space after blowing up procedure. These fractional sheaves are characterized by D0, D2 and D4 Ramon-Ramon charges, which have special type, calculated for C^3/Z_3 case. BPS central charge for C^3/Z_3 orbifold is calculated through Ramon-Ramon charges and Picard-Fuchs periods.

Key words: *super-symmetry algebra, central charge, noncompact manifolds, orbifold points, coherent sheaves, Euler characteristic.*

Introduction

Modern high energy theoretical physics is a unified theory of all particles and all interactions. It is Theory of Everything, because it gives a universal description of the processes occurring on modern accelerators, and processes in the Universe.

Theory of everything (abbr. TOE) - hypothetical combined physical and mathematical theory describing all known fundamental interactions. This theory unifies all four fundamental interactions in nature. The main problem of building TOE is that quantum mechanics and general theory of relativity have different applications. Quantum mechanics is mainly used to describe the microword, and general relativity is applicable to the macro world. But it does not mean that such theory cannot be constructed.

Modern physics requires from TOE the unification of four fundamental interactions:

- gravitational interaction;
- electromagnetic interaction;
- strong nuclear interaction;
- weak nuclear interaction.

The first step towards this was the unification of the electromagnetic and weak interactions in the theory of electro-weak interaction created by in 1967 by Stephen Weinberg, Sheldon Glashow and Abdus Salam. In 1973, the theory of strong interaction was proposed.

The main candidate as TOE is F-theory, which operates with a large number of dimensions. Thanks to the ideas of Kaluza and Klein it became possible to create theories operating with large extra dimensions. The use of extra dimensions prompted the answer to the question about why the effect of gravity appears much weaker than other types of interactions. The generally accepted answer is that gravity exists in extra dimensions, therefore its effect on observable measurements weakened.

F-theory is a string twelve-dimensional theory defined on energy scale of about 10^{19} GeV [1]. F-theory compactification leads to a new type of vacuum, so to study supersymmetry we must compactify the F-theory on Calabi-Yau manifolds. Since there are many Calabi-Yau manifolds, we are dealing with a large number of new models implemented in low-energy approximation. Studying the singularities of Calabi manifold determines the physical characteristics of topological solitonic states which plays the role of particles in high energy physics.

Compactification of F-theory on different Calabi-Yau manifolds allows to calculate topological invariants.

Let us consider in more detail the compactification of F-theory on threefolds Calabi Yau.

1. Calabi-Yau threefold compactification

Twelve-dimensional space describing space-time and internal degrees of freedom, we compactify as follows:

$$R^6 \times X^6,$$

where R^6 - six-dimensional space-time, on which acts conformal group $SO(4, 2)$, and X^6 - threefold, which is three-dimensional Calabi Yau complex manifold [2].

2. Toric representation of threefolds

Let's consider weighted projective space defined as follows:

$$P_{\omega_1, \dots, \omega_5}^4 = P^4 / Z_{\omega_1} \times \dots \times Z_{\omega_5},$$

where P^4 - four-dimensional projective space, Z_{ω_i} - cyclic group of order ω_i . On weighted projective space $P_{\omega_1, \dots, \omega_5}^4$ is defined polynomial $W(\varphi_1, \dots, \varphi_5)$, called superpotential which satisfies the homogeneity condition

$$W(x^{\omega_1} \varphi_1, \dots, x^{\omega_5} \varphi_5) = x^d W(\varphi_1, \dots, \varphi_5),$$

where $d = \sum_{i=1}^5 \omega_i$, $\varphi_1, \dots, \varphi_5 \in P_{\omega_1, \dots, \omega_5}^4$. The set of points $p \in P_{\omega_1, \dots, \omega_5}^4$, satisfying the condition $W(p) = 0$ forms Calabi-Yau threefold $X_d(\omega_1, \dots, \omega_5)$.

The simplest examples of toric varieties [3] are projective spaces. Let's consider P^2 defined as follows:

$$P^2 = \frac{C^3/0}{C/0},$$

where dividing by $C/0$ means identification of points connected by equivalence relation

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$$

$$\lambda \in C/0,$$

x, y, z are homogeneous coordinates. Elliptic curve in P^2 is described by the Weierstrass equation

$$y^2 z = x^3 + axz^2 + bz^3.$$

In general Calabi-Yau manifold can be described by Weierstrass form

$$y^2 = x^3 + xf + g,$$

which describes an elliptic fibration (parametrized by (y, x)) over the base, where f, g - functions defined on the base. In some divisors D_i the layer are degenerated. Such divisors are zeros of discriminant

$$\Delta = 4f^3 + 27g^2.$$

The singularities of Calabi-Yau manifold are singularities of its elliptic fibrations. These singularities are coded in polynomials f, g and their type determines the gauge group and matter content of compactified F-theory.

The classification of singularities of elliptic fibrations was given by Kodaira and presented table 1.

Table 1. Kodaira classification of singularities of elliptic fibrations

$ord(\Delta)$	Type of fiber	Type of singularity
0	smooth	no
n	I_n	A_{n-1}
2	II	no
3	III	A_1
4	IV	A_2
$n+6$	I_n^*	D_{n+4}
8	IV^*	E_6
9	III^*	E_7
10	II^*	E_8

The classification of elliptic fibers is presented in Figure 1.

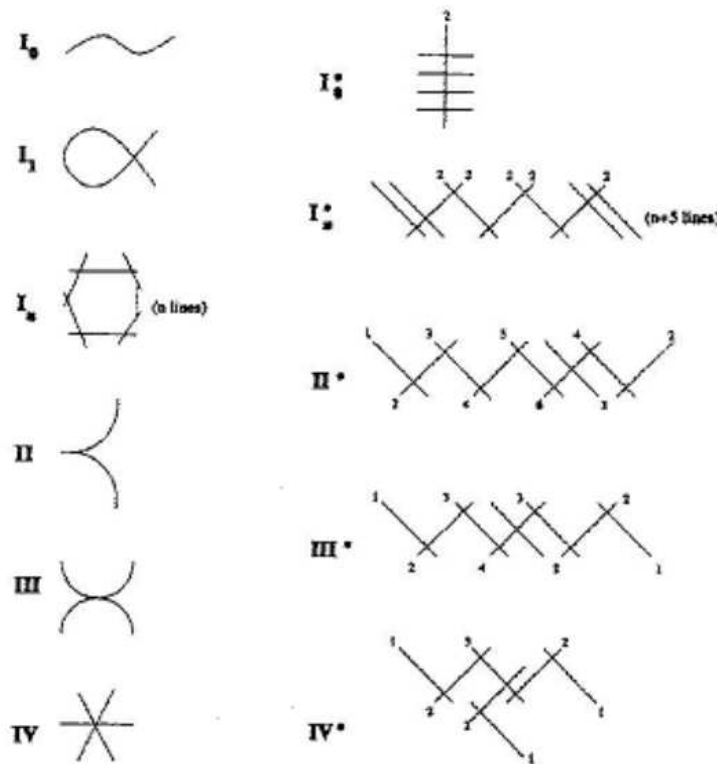


Рис. 1: Fig.1. The classification of elliptic fibers.

3. Calculation of topological invariants

4. Ramon-Ramon charges

One of the most interesting problems of modern high-energy physics is the calculation of topological invariants — analogs of high-energy observables in physics. In this aspect,

symmetries and the use of the apparatus of algebraic geometry play an indispensable role. We considered orbifold as simplest non-flat constructions. For D3-branes on such internal space C^n/Γ the representations are characterized by gauge groups $G = \oplus_i U(N_i)$. In this case the superpotential is of N=4 $U(N)$ super Yang-Mills,

$$W_{N=4} = \text{tr} X^1 [X^2, X^3],$$

where X^i are chiral matter fields in production of fundamental representation $V^i \cong C^{N_i}$ of the group $U(N_i)$. Blow up modes of orbifold singularities can be considered as coordinates of complexified Kahler moduli space. Quiver diagrams are used for describing D-branes near orbifold point.

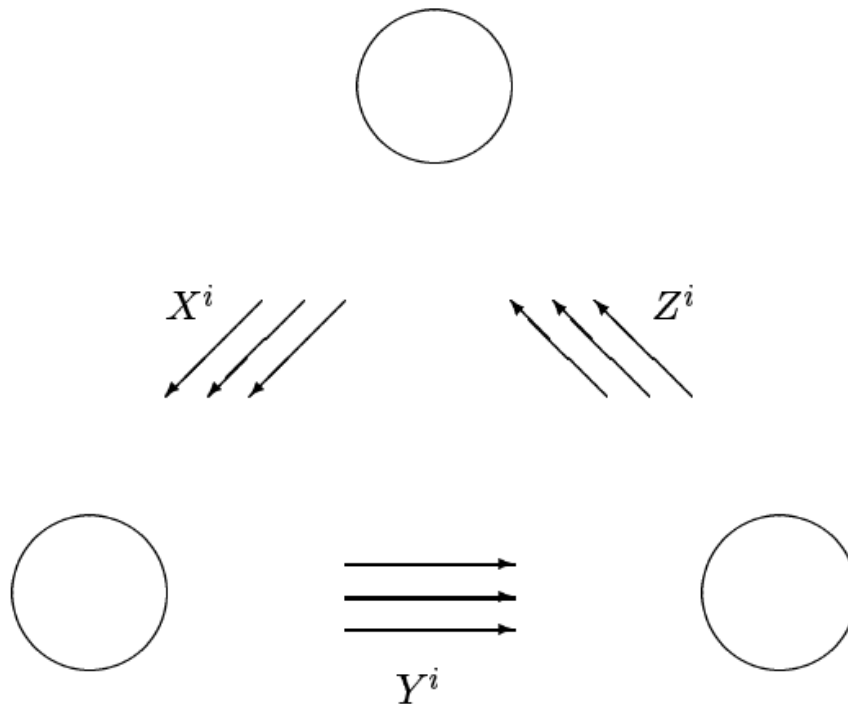


Рис. 2: Fig.2. The C^3/Z_3 quiver.

In this case it is possible to calculate Euler character defined as

$$\chi(A, B) = \sum_i (-1)^i \dim \text{Ext}^i(A, B),$$

where $\text{Ext}^0(A, B) \equiv \text{Hom}(A, B)$ and A, B are coherent sheaves over projective space, P^N (general case), which are representations of orbifold space after blowing up procedure.

Since we will deal with orbifolds C^3/Z_3 in the future, it is necessary to emphasize the following equivalence relation

$$(x_1 x_2 x_3) \sim (e^{2i\pi/3} x_1, e^{2i\pi/3} x_2, e^{2i\pi/3} x_3), e^{2i\pi/3} \in Z_3$$

Orbifold is not a manifold, since it has singularities at a point $(0, 0, 0)$. Blowing up the singularity of the orbifold C^3/Z_3 , we obtain a sheaf $\mathcal{O}_{P^2}(-3)$ with which we will work further.

In particular, the Euler matrix for sheaves \mathcal{O}_{P^2} , $\mathcal{O}_{P^2}(1)$, $\mathcal{O}_{P^2}(2)$ over projective space, P^2 looks like

$$\chi(\mathcal{O}_{P^2}(1), \mathcal{O}_{P^2}(2)) = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Transposed matrix has the form

$$\begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 3 & 1 \end{pmatrix}.$$

The rows of matrices are RR-charges characterizing the sheaves:

$$\mathcal{O}_{P^2}(-3) = (6 \ 3 \ 1), \mathcal{O}_{P^2}(-2) = (3 \ 1 \ 0), \mathcal{O}_{P^2}(-1) = (1 \ 0 \ 0), \quad (1)$$

$$\mathcal{O}_{P^2} = (0 \ 0 \ 1), \mathcal{O}_{P^2}(1) = (0 \ 1 \ 3), \mathcal{O}_{P^2}(2) = (1 \ 3 \ 6), \quad (2)$$

which can be written through large volume charges (Q_4, Q_2, Q_0) :

$$Q_4 = n_1 - 2n_2 + n_3, \quad Q_2 = -n_1 + n_2, \quad Q_0 = \frac{n_1 + n_2}{2}$$

included in the definition of the Chern character $ch(n_1 n_2 n_3)$

$$ch(n_1 n_2 n_3) = Q_4 + Q_2 w + Q_0 w^2,$$

where w - Wu number. Then sheaves (1), (2) describe fractional branes [4]

$$\mathcal{O}_{P^2}(-3) = (1 \ -3 \ \frac{9}{2}), \mathcal{O}_{P^2}(-2) = (1 \ -2 \ \frac{4}{2}), \mathcal{O}_{P^2}(-1) = (1 \ -1 \ \frac{1}{2}),$$

$$\mathcal{O}_{P^2} = (1 \ 0 \ 0), \mathcal{O}_{P^2}(1) = (1 \ 1 \ \frac{1}{2}), \mathcal{O}_{P^2}(2) = (1 \ 2 \ \frac{4}{2}),$$

General formula for Chern character of bundle E :

$$ch(E) = k + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \dots,$$

where $c_i(E)$ are the Chern classes of line bundle E . In our case of a line bundle $\mathcal{O}_{P^2}(k)$, only the first Chern class is nonzero, and therefore the formula for the Chern character is following

$$ch(E) = k + c_1(E) + \frac{1}{2}c_1^2 \quad (3)$$

As

$$1 + c_1(E) + \dots + c_n(E) = \prod_{i=1}^n (1 + w_i),$$

then $c_1(E) = w_1 = w$ and formula (3) can be rewritten

$$ch(n_1 n_2 n_3) = Q_4 + Q_2 w + Q_0 w^2, \quad (4)$$

where Ramon-Ramon charges $(n_1 n_2 n_3)$ characterize the bundle E , the rank of the line bundle $Q_4 = 1, Q_2 = c_1$ by the fundamental cycle, $Q_0 = \frac{c_1^2}{2}$ from a comparison of formulas (3) and (4).

Thus fractional sheaves $\mathcal{O}_{P^2}(k)$ are characterized by Q_0, Q_2, Q_4 Ramon-Ramon charges, which have special type, calculated for C^3/Z_3 case.

5. BPS central charge

As we are interested in the moduli spaces, we give them a visual definition. Suppose we have a cube curve with the parameter λ

$$y^2 - x(x - 1)(x - \lambda) = 0 \tag{5}$$

As λ - the variable value, then the equation (5) describes a continuous family of cubic curves. The parameter spaces describing continuous families of manifolds are called moduli spaces. We form $\frac{dx}{y}$ the form where y are determined from equation (5). It turns out that periods $\pi_1(\lambda), \pi_2(\lambda)$:

$$\pi_1(\lambda) = 2 \int_0^1 \frac{dx}{[x(x - 1)(x - \lambda)]^{1/2}}, \quad \pi_2(\lambda) = 2 \int_1^\lambda \frac{dx}{[x(x - 1)(x - \lambda)]^{1/2}}$$

satisfy Picard-Fuchs equation

$$\frac{1}{4}\pi_i + (2\lambda - 1)\frac{d\pi_i}{d\lambda} + \lambda(\lambda - 1)\frac{d^2\pi_i}{d\lambda^2} = 0. \tag{6}$$

Periods that satisfy equation (6) describe the moduli space of a cubic curves. For the moduli space of a line bundle $\mathcal{O}_{P^2}(-3)$, Picard-Fuchs equation and its solutions are written as

$$\left(z \frac{d}{dz}\right)^3 + 27z \left(z \frac{d}{dz}\right) \left(z \frac{d}{dz} + \frac{1}{3}\right) \left(z \frac{d}{dz} + \frac{2}{3}\right) \Pi = 0$$

$$\Pi_0 = 1,$$

$$\Pi_1 = \frac{1}{2i\pi} \log z = t = w_0,$$

$$\Pi_2 = t^2 - t - \frac{1}{6} = -\frac{2}{3}(w_0 - w_1).$$

The BPS central charge [5] associated with the D-brane over C^3/Z_3 with Ramon-Ramon-charge $n = (n_1 n_2 n_3)$ and with the Picard-Fuchs period $\Pi = (\Pi_0 \Pi_1 \Pi_2)$ is given by the formula

$$Z(n) = n \cdot \Pi$$

The central charge associated with the sheave $\mathcal{O}_{P^2}(k)$ is given by the formula

$$Z(\mathcal{O}_{P^2}(k)) = -(k + \frac{1}{3}w_0) + \frac{1}{3}w_1 + \frac{1}{2}k^2 + \frac{1}{2}k + \frac{1}{3}.$$

Conclusion

In the framework of F-theory we presented the ideology of extra dimensional spaces. It was stressed the exceptional role of topological invariants for Calabi-Yau manifolds. We have considered the special type of the space of extra dimensions - orbifold C/Z_3 . Using blowing up procedure of singularity we calculated special type of topological invariant - Ramon-Ramon central charges of fractional sheaves, in which is encoded the information about the structure of line bundles. Consideration of moduli space of orbifold leads us to the equation of Picard-Fuchs periods, through which we calculated central charge for sheave $\mathcal{O}_{P^2}(k)$. This topological invariant is of importance because of information of stability of D-branes as bound states of fractional branes or sheaves presented in this paper.

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ЦЕНТРАЛЬНИЙ ЗАРЯД І ТОПОЛОГІЧНИЙ ІНВАРІАНТ МНОГОВИДУ КАЛАБІ-ЯУ

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РЕЗЮМЕ

F-теорія, як дванадцятивимірною теорією, яка є претендентом на Теорію всього, повинна бути компактифікованою на еліптично розшарований трифолд або фоурфолд Калабі-Яу. Такі многовиди мають еліптичну криву як шар і їх бази можуть мати особливості. Ми розглядали орбіфолд як найпростішу неплоску конструкцію. Роздуті моди сингулярностей орбіфолду можуть бути розглянуті як координати комплексифікованого простору модулів Келера. Для опису D-бран поблизу орбіфолду використовуються діаграми квівера. У цьому випадку можна обчислити характеристику Ейлера, визначену через групи $\text{Ext}^i(A, B)$ і когерентні пучки A, B над проективним простором, які є представленнями орбіфолду після процедури роздуття. Ці дробові пучки характеризуються D0, D2 і D4 Рамон-Рамон зарядами, які мають спеціальний тип, розрахований для S^3/Z_3 простору. BPS центральний заряд для S^3/Z_3 орбіфолду обчислено через заряди Рамон-Рамона і періоди Пікара-Фукса.

Ключові слова: алгебра суперсиметрії, центральний заряд, некомпактні многовиди, точки орбіфолду, когерентні пучки, характеристика Ейлера.

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ЦЕНТРАЛЬНИЙ ЗАРЯД И ТОПОЛОГИЧЕСКИЙ ИНВАРИАНТ МНОГООБРАЗИЯ КАЛАБИ-ЯУ

РЕЗЮМЕ

F-теория, как двенадцатимерная теория, являющаяся претендентом на Теорию Всего, должна быть компактифицирована на эллиптически расслоенный трифолд или фоурфолд

Калаби-Яу. Такие многообразия имеют эллиптическую кривую в качестве слоя и их базы могут иметь особенности. Мы рассматривали орбифолд как простейшую неплоскую конструкцию. Раздутые моды сингулярностей орбифолда могут быть рассмотрены как координаты комплексифицированного пространства модулей Кэлера. Для описания D-бран около орбифолда используются диаграммы квивера. В этом случае можно вычислить характеристику Эйлера, определенную через группы $\text{Ext}^i(A, B)$ и когерентные пучки A, B над проективным пространством, которые являются представлением пространства орбифолда после процедуры раздутия. Эти дробные пучки характеризуются D0, D2 и D4 Рамон-Рамон зарядами, которые имеют специальный тип, рассчитанный для S^3/Z_3 пространства. BPS центральный заряд для S^3/Z_3 орбифолда вычислен через заряды Рамон-Рамона и периоды Пикара-Фукса.

Ключевые слова: алгебра суперсимметрии, центральный заряд, некомпактные многообразия, точки орбифолда, когерентные пучки, эйлерова характеристика.