The main purpose of this article as well as the previous one [14] is classification of ternary quasigroups according to their parastrophic symmetry groups. Since each of these groups is a subgroup of the symmetric group of degree 4, i.e. $S_4$, and parastrophic symmetry groups of parastrophic quasigroups are conjugate, then quasigroups whose parastrophic symmetry groups are not pairwise conjugated are considered in these articles. The list of all different parastrophes, sets of identities which define the corresponding varieties, canonical decompositions of ternary group isotopes belonging to these varieties is given.

Key words: ternary quasigroup, variety of a quasigroup, parastrophic quasigroups, parastrophic symmetry, parastrophic varieties, parastrophic symmetry groups.

Introduction

One can find all notions and results in [14]. Here we give necessary information for reading only this article.

In [14], it is proved that a class of quasigroups whose parastrophic symmetry groups contain the given subgroup of the group $S_4$ forms a variety. For example, the trivial subgroup $E := \{e\}$ defines the variety of all ternary quasigroups. The quasigroup varieties are parastrophic if and only if the corresponding subgroups are conjugate.

The set $\text{Sub}(S_4)$ of all subgroups of the symmetric group $S_4$ has 30 elements. The mapping $H \mapsto \tau H \tau^{-1}$ is an action of $S_4$ on $\text{Sub}(S_4)$. A full set of representatives of the corresponding orbits on $\text{Sub}(S_4)$ is selected. The list contains 11 elements: $E$, $S_2$, $S_{22}$, $A_3$, $Z_4$, $K_4$, $C_4$, $S_3$, $D_8$, $A_4$, $S_4$.

In [14], sets of identities defining such varieties are deduced for all subgroups $A_3$, $D_8$, $S_3$, $A_4$, $S_4$. The necessary and sufficient conditions for a ternary group isotope to belong to each of these varieties are described. The list of different parastrophes is found for quasigroups whose parastrophic symmetry group is equal to the given subgroup of $S_4$.

In the current article, these results are obtained for the rest of subgroups: $S_2$, $S_{22}$, $Z_4$, $K_4$, $C_4$.

Preliminaries

A ternary operation $f$ is called invertible if there exist three operations $(14)f$, $(24)f$, $(34)f$ called inverses or divisions such that the identities

\begin{align}
    f((14)f(x, y, z), y, z) &= x, \quad (1) \\
    f(x, (24)f(x, y, z), z) &= y, \quad (2) \\
    f(x, y, (34)f(x, y, z)) &= z, \quad (3)
\end{align}

and

\begin{align}
    (14)f(f(x, y, z), y, z) &= x, \quad (4) \\
    (24)f(x, f(x, y, z), z) &= y, \quad (5) \\
    (34)f(x, y, f(x, y, z)) &= z \quad (6)
\end{align}

hold. In this case, the algebra $(Q; f, (14)f, (24)f, (34)f)$ (in brief, $(Q; f)$) is called a ternary quasigroup [2]. It is easy to verify that all divisions of an invertible operation are also invertible and
so are their divisions. All of these operations having such connections with the main operation f are called parastrophes of f. Namely, a $\sigma$-parastrophe of an invertible operation f is called an operation $\sigma f$ defined by $\sigma f(x_1, x_2, x_3) = x_{4\sigma} \iff f(x_1, x_2, x_3) = x_4, \; \sigma \in S_4$, where $S_4$ denotes the group of all bijections of the set \{0, 1, 2, 3\}. Therefore in general, every invertible operation has 24 parastrophes. Some of them can coincide. Sometimes it is convenient to use an equivalent form of the formula:

$$\sigma f(x_1, x_2, x_3) = x_4 \iff f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}) = x_{4\sigma^{-1}}, \; \sigma \in S_4. \quad (7)$$

If $4\sigma = 4$, the parastrophe is called principal and can be found by

$$\sigma f(x_1, x_2, x_3) = f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}), \; \sigma \in S_4. \quad (8)$$

Since for every invertible operation f and for every permutation $\sigma \in S_4$ the relations

$$\sigma(\tau f) = \sigma \tau f \quad \text{and} \quad f = f \quad (9)$$

hold, then the symmetric group $S_4$ defines an action on the set $\Delta_3$ of all ternary invertible operations defined on the same carrier. In particular, the fact implies that the number of different parastrophes of an invertible operation is a factor of 24. More precisely, it is equal to $24/|\text{Ps}(f)|$, where $\text{Ps}(f)$ denotes a stabilizer group of f under the action called parastrophic symmetry group of the operation f.

**Symmetric group** $S_4$. Let $S_n$ denote the symmetric group of degree n, i.e., the group of all bijections from the set \{1, \ldots, n\} onto itself. Therefore, $S_4$ has 24 members:

$$S_4 = \{\sigma, (04)\sigma, (14)\sigma, (24)\sigma, (34)\sigma \mid \sigma \in S_3\} = \{\sigma, \sigma(04), \sigma(14), \sigma(24), \sigma(34) \mid \sigma \in S_3\}. \quad (10)$$

The equalities (4) and (10) imply that every parastrophe of an invertible operation f is a principal parastrophe of a division of f and equals a division of a principal parastrophe of f.

The set $\text{Sub}(S_4)$ of all subgroups of the symmetric group $S_4$ has 30 elements. The mapping $H \mapsto \tau H \tau^{-1}$ is an action of $S_4$ on $\text{Sub}(S_4)$. A full set of representatives of the corresponding orbits on $\text{Sub}(S_4)$ is the following and it contains 11 elements:

$$S_4, \quad E := \{\iota\}, \quad S_2 := \{\iota, (12)\}, \quad S_22 := \{\iota, (12)(34)\}, \quad A_3 := \{\iota, (123), (132)\}, \quad Z_4 := \{\iota, (12)(34), (1423), (1324)\}, \quad K_4 := \{\iota, (12)(34), (13)(24), (14)(23)\}, \quad C_4 := \{\iota, (12), (34), (12)(34)\}, \quad S_3 := \{(12), (13), (23), (123), (132)\}, \quad D_8 := \{\iota, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\},$$

$$A_4 := \{(123), (132), (13)(14)(132), (124), (234), (243), (13)(24), (12)(34), (14)(23)\}.$$

An operation f is called:

- **symmetric**, if some of its parastrophes coincide, i.e. $\text{Ps}(f) \neq \{\iota\}$;
- **asymmetric**, if all its parastrophes are pairwise different, i.e. $\text{Ps}(f) = \{\iota\}$;
- **totally symmetric**, if all its parastrophes coincide, i.e. $\text{Ps}(f) = S_{n+1}$;
- **commutative**, if all its principal parastrophes coincide, i.e. $\text{Ps}(f) \supseteq S_n$;
- **semisymmetric**, if f has at most two different parastrophes, i.e. $\text{Ps}(f) \supseteq A_{n+1}$.
- **dihedrally symmetric** ($n=3$), if f has at most three different parastrophes, i.e. $\text{Ps}(f) \supseteq D_8$.
1. Group isotopes

A ternary groupoid \((Q; f)\) is called a group isotope, if there exists a group \((G; \cdot)\) and bijections \(\alpha, \beta, \gamma, \delta\) from \(Q\) to \(G\) such that

\[
f(x, y, z) = \delta^{-1}(\alpha x \cdot \beta y \cdot \gamma z)
\]

for all \(x, y, z\) in \(Q\).

**Definition 1.** Let \((Q; f)\) be a ternary group isotope and let \((Q; +, 0)\) be a group, \(\alpha_1, \alpha_2, \alpha_3\) be its bijections with \(\alpha_10 = \alpha_20 = \alpha_30 = 0\) and \(a \in Q\). If

\[
f(x_1, x_2, x_3) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a,
\]

then the tuple \((+, \alpha_1, \alpha_2, \alpha_3, a)\) is called a 0-canonical decomposition of \((Q; f); (Q; +)\) the canonical decomposition group; \(\alpha_1, \alpha_2, \alpha_3\) its coefficients; \(a\) is called a free member.


**Lemma 1.** [3] Let \((Q; f)\) be an arbitrary ternary group isotope and let (1) be its canonical decomposition. Then its divisions and principal paraphrases are

\[
\begin{align*}
(1) f(x_1, x_2, x_3) &= \alpha_1^{-1}(x_1 - a - \alpha_3 x_3 - \alpha_2 x_2), \\
(2) f(x_1, x_2, x_3) &= \alpha_2^{-1}(-\alpha_1 x_1 + x_2 - a - \alpha_3 x_3), \\
(3) f(x_1, x_2, x_3) &= \alpha_3^{-1}(-\alpha_2 x_2 - \alpha_1 x_1 + x_3 - a), \\
(11) f(x_1, x_2, x_3) &= \alpha_1 x_{1\sigma^{-1}} + \alpha_2 x_{2\sigma^{-1}} + \alpha_3 x_{3\sigma^{-1}} + a, \quad \sigma \in S_3.
\end{align*}
\]

A pair \((Q; \Omega)\) is called a quasigroup algebra, if \(Q\) is a set and \(\Omega\) is a set of invertible operations defined on \(Q\). A variable is called quadratic in an equality, if it has exactly two appearances in it. Let \(\omega\) be a term, then \(\omega(x)\) means that \(x\) has an appearance in \(\omega\).

**Lemma 2.** [4, Theorem 3] Let \((Q; \cdot; \Omega)\) be a quasigroup algebra, additionally, \((Q; \cdot)\) is a group isotope. If the algebra satisfies an identity

\[
\omega_1(x) \cdot \omega_2(y) = \omega_3(y) \cdot \omega_4(x),
\]

where the variables \(x\) and \(y\) are quadratic, then an arbitrary group being isotopic to \((Q; \cdot)\) is commutative.

**Lemma 3.** [4, Theorem 3] Let \((Q; +; \Omega)\) be a quasigroup algebra, additionally, \((Q; +, 0)\) is a group, \(\alpha \in \Omega\) and \(\alpha 0 = 0\). If the algebra satisfies an identity

\[
\alpha(\omega_1(x) + \omega_2(y)) = \omega_3(z) + \omega_4(u),
\]

then \(\alpha\) is either an automorphism of \((Q, +)\), if \(z = x, u = y\) or anti-automorphism of \((Q, +)\), if \(z = y, u = x\).
2. Quasigroups with a fixed parastrophic symmetry group

Let $\mathcal{P}_Q(H)$ and $\mathcal{P}_s(Q)(H)$ denote the sets of all ternary invertible operations defined on a set $Q$ whose parastrophic symmetry group respectively contains the group $H \in S_4$ and equals the group $H$. Let $\mathcal{P}(H)$ and $\mathcal{P}_s(H)$ denote the class of all quasigroups whose parastrophic symmetry group includes the group $H \in S_4$ and equals the group $H$ respectively. It is easy to see that

- a ternary quasigroup $(Q; f)$ belongs to the class $\mathcal{P}(H)$ if and only if $\sigma f = f$ for all $\sigma$ from a set of generators of the group $H$ therefore, the class of quasigroups $\mathcal{P}(H)$ is a variety;
- the set $\{\mathcal{P}_s(Q)(H) \mid H$ is a subgroup of $S_4\}$ is a partition of the set $\Delta_3(Q)$ of all ternary invertible operations defined on $Q$.

Lemma 4. If a non-trivial principal parastrophe of a ternary group isotope $f$ coincides with $f$, then its canonical decomposition group is commutative.

3. The group $S_{22}$.

One of generator sets of the group $S_{22}$ is $\{(12)(34)\}$. Therefore, the variety $\mathcal{P}(S_{22})$ is defined by the identity

$$^{(12)(34)}f = f.$$ (13)

The equality means that for all $x_1, x_2, x_3$

$$^{(12)(34)}f(x_1, x_2, x_3) = f(x_1, x_2, x_3).$$ (14)

Since $^{(12)(34)}f = ^{(12)}(34)f$, then

$$^{(34)}f(x_2, x_1, x_3) = f(x_1, x_2, x_3).$$ (14)

Therefrom,

$$f(x_2, x_1, f(x_1, x_2, x_3)) = x_3.$$ (14)

Thus, the following assertion is true.

Proposition 1. A ternary quasigroup $(Q; f)$ belongs to the variety $\mathcal{P}(S_{22})$ if and only if

$$f(y, x, f(x, y, z)) = z.$$ (15)

Proposition 2. Let $(Q; f)$ be a ternary quasigroup. If $\mathcal{P}_s(f) = S_{22}$, then $f$, $^{(12)}f$, $^{(13)}f$, $^{(14)}f$, $^{(24)}f$, $^{(23)}f$, $^{(123)}f$, $^{(132)}f$, $^{(124)}f$, $^{(142)}f$, $^{(1324)}f$, $^{(13)(24)}f$ are all different parastrophes of the operation $f$.

Proof. One can verify that

$$S_4/S_{22} = \{ S_{22}, (12)S_{22}, (13)S_{22}, (14)S_{22}, (12)(34)S_{22}, (23)S_{22}, (132)S_{22}, (13)(24)S_{22}, (12)(34)S_{22} \}$$

then $\iota$, $(12)$, $(13)$, $(14)$, $(24)$, $(23)$, $(123)$, $(132)$, $(124)$, $(142)$, $(1324)$, $(13)(24)$ are all representatives from $S_4/S_{22}$. It remains to apply Theorem 3 [14].

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Theorem 2. A ternary group isotope \((Q, f)\) belongs to \(Ψ(S_{22})\) if and only if there exists a group \((Q, +, 0)\), its automorphism \(β\), a bijection \(α\) and an element \(a \in Q\) such that \(β^2 = i\), \(α0 = 0\), \(-βa = a\) and
\[
f(x_1, x_2, x_3) = αx_1 - βαx_2 + βx_3 + a.
\]

Proof. Let \(f(x_1, x_2, x_3) = α_1x_1 + α_2x_2 + α_3x_3 + a\) be a 0-canonical decomposition of \(f\). Using (12), the identity (14) can be written as follows:
\[
α^{-1}_3(-α_2x_1 - α_1x_2 + x_3 - a) = α_1x_1 + α_2x_2 + α_3x_3 + a.
\]
Let \(β := α_3\) and \(α := α_1\), then Lemma 3. implies that \(β\) is an automorphism of the group \((Q; +, 0)\) and
\[
-β^{-1}α_2 x_1 - β^{-1}αx_2 + β^{-1}x_3 - β^{-1}a = αx_1 + α_2x_2 + βx_3 + a.
\]
Taking into account the uniqueness of a canonical decomposition, we obtain the identity equivalent to
\[
-β^{-1}α_2 = α, \quad -β^{-1}α = α_2, \quad β^{-1} = β, \quad -β^{-1}a = a.
\]
The first and the second equality imply \(-βα = -β^{-1}α\) that is, \(β = β^{-1}\) therefore, \(β^2 = i\). But the third equality means that \(β^2 = i\). Consequently, the identity (17) is equivalent to
\[
β^2 = i, \quad α_2 = -βα, \quad βa = a.
\]

4. The symmetry group contains \(Z_4\).
At first, we find an identity which describes the variety \(Ψ(Z_4)\).

Proposition 3. A ternary quasigroup \((Q, f)\) belongs to \(Ψ(Z_4)\) if and only if it satisfies the identity
\[
f(x, f(y, z, x), z) = y.
\]

Proof. Since \((1423)\) generates the subgroup \(Z_4\) of the symmetric group \(S_4\), then \(Z_4\) is a subgroup of the symmetry group of a ternary quasigroup \((Q, f)\) if and only if the identity
\[
(1423)f = f
\]
holds in \((Q, f)\). Because \((1423) = (123)(24)\) and \((123)^{-1} = (132)\), then
\[
(1423)f = (123)(24)f = (123)(24)f.
\]
Taking into account (2) and \((123)^{-1} = (132)\), we get
\[
(1423)f(x_1, x_2, x_3) = (123)(24)f(x_1, x_2, x_3) = (24)f(x_3, x_1, x_2).
\]
Then the equality (19) can be written as
\[
(24)f(x_3, x_1, x_2) = f(x_1, x_2, x_3).
\]
Using the definition of \((24)\)-division, we have
\[
f(x_3, f(x_1, x_2, x_3), x_2) = x_1.
\]

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Proposition 4. Let \((Q; f)\) be a ternary quasigroup. If \(\Psi(f) = Z_4\), then \(f, (12)f, (13)f, (23)f, (14)f, (24)f\) are all different parastrophes of the operation \(f\).

Proof. Since
\[S_4 = Z_4 \sqcup (12)Z_4 \sqcup (13)Z_4 \sqcup (23)Z_4 \sqcup (14)Z_4 \sqcup (24)Z_4,\]
then \(\iota, (12), (13), (23), (14), (24)\) are all representatives from \(S_4/Z_4\). \(\square\)

Theorem 3. A ternary group isotope \((Q, f)\) belongs to \(\Psi(Z_4)\) if and only if there exists an abelian group \((Q, +, 0)\), its automorphism \(\alpha\) and an element \(a \in Q\) such that \(\alpha^4 = \iota, \alpha^3a = -a\) and
\[f(x_1, x_2, x_3) = \alpha x_1 + \alpha^3 x_2 - \alpha^2 x_3 + a.\] (21)

Proof. Let \((Q, f)\) be a ternary group isotope and \((1)\) be its 0-decomposition. As it was shown above, the quasigroup \((Q, f)\) belongs to the variety \(\Psi(Z_4)\) if and only if \((20)\) holds. Lemma 1. implies that the identity can be written as
\[\alpha_2^{-1}(-\alpha_3x_3 + x_1 - a - \alpha_3x_2) = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + a.\]
Fixing \(x_3\) and \(x_2\) one by one in turn and applying Lemma 3., we obtain \(\alpha_2^{-1}\) which is both an automorphism and anti-automorphism. Consequently, the group \((Q, +)\) is commutative. Apply \(\alpha_2\) to both sides of the identity:
\[x_1 - \alpha_3x_2 - \alpha_1x_3 - a = \alpha_2\alpha_1x_1 + \alpha_2^3x_2 + \alpha_2\alpha_3x_3 + \alpha_2a.\]
The left and the right sides of the equality are 0-canonical decomposition of the same operation. Therefore according to Theorem 1., the corresponding coefficients and the free members are equal.
\[\iota = \alpha_2\alpha_1, \quad -\alpha_3 = \alpha_2^2, \quad -\alpha_1 = \alpha_2\alpha_3, \quad -a = \alpha_2a.\]
These equalities are equivalent to the equalities
\[\alpha_2 = \alpha_1^{-1}, \quad \alpha_3 = -\alpha_1^{-2}, \quad -\alpha_1 = -\alpha_1^{-1}\alpha_1^{-2}, \quad \alpha_2a = -a.\]
Let \(\alpha := \alpha_1\). Use the first and the second equalities to form the third one:
\[\alpha_2 = \alpha^{-1} = \alpha^3, \quad \alpha_3 = -\alpha^2 = -\alpha^2, \quad \alpha_4 = \iota, \quad \alpha^3a = -a.\]
The theorem has been proved. \(\square\)

5. The symmetry group contains \(K_4\).

Proposition 5. A ternary quasigroup \((Q, f)\) belongs to \(\Psi(K_4)\) if and only if it satisfies the identities
\[f(y, x, f(x, y, z)) = z, \quad f(z, f(x, y, z), x) = y.\] (22)

Proof. Since the permutations \((12)(34), (13)(24)\) generate the subgroup \(K_4\) of the symmetric group \(S_4\), then \(K_4\) is a subgroup of the symmetry group of a ternary quasigroup \((Q, f)\) if and only if the equalities
\[(12)(34)f = f, \quad (13)(24)f = f\] (23)
Therefore, the equalities (23) can be written as identities in the quasigroup \((Q, f)\):

\[
(34)f(x_2, x_1, x_3) = f(x_1, x_2, x_3), \quad (24)f(x_3, x_2, x_1) = (24)f(x_3, x_2, x_1).
\]

Applying the definition of the divisions, we obtain

\[
f(x_2, x_1, f(x_1, x_2, x_3)) = x_3, \quad f(x_3, f(x_1, x_2, x_3), x_1) = x_2.
\]

\[\square\]

**Proposition 6.** Let \((Q, f)\) be a ternary quasigroup. If \(\text{Ps}(f) = K_4\), then \(f, \ (12)f, \ (13)f, \ (14)f, \ (132)f, \ (123)f\) are all different parastrophes of the operation \(f\).

**Proof.** Since

\[
S_4 = K_4 \sqcup (12)K_4 \sqcup (13)K_4 \sqcup (14)K_4 \sqcup (132)K_4 \sqcup (123)K_4,
\]

then \(\iota, \ (12), \ (13), \ (14), \ (132), \ (123)\) are all representatives from \(S_4/Z_4\).

\[\square\]

**Theorem 4.** A ternary group isotope \((Q, f)\) belongs to \(\Psi(K_4)\) if and only if there exists a group \((Q, +, 0)\), its involuting automorphisms \(\alpha, \beta\) and an element \(a \in Q\) such that \(\alpha a = \beta a = -a\), \(\beta \alpha = I_a \alpha \beta\) and

\[
f(x, y, z) = -\beta \alpha x + \alpha y + \beta z + a. \quad (25)
\]

**Proof.** Let \((Q, f)\) be a ternary group isotope and \(1\) be its 0-decomposition. As it was shown above, the quasigroup \((Q, f)\) belongs to the variety \(\Psi(Z_4)\) if and only if (24) holds. Lemma 1 implies that these identities can be written as

\[
\alpha^{-1}_3(-\alpha_2 x_1 - \alpha_1 x_2 + x_3 - a) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a, \quad \alpha_2^{-1}(\alpha_1 x_3 + x_2 - a - \alpha_3 x_1) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a.
\]

\(\alpha_3\) is an automorphism and \(\alpha_2\) is an anti-automorphism of the group \((Q,+)\) as the result of fixing \(x_3\) and applying Lemma 3. Apply \(\alpha_3\) to the first identity and \(\alpha_2\) to the second one:

\[
-\alpha_2 x_1 - \alpha_1 x_2 + x_3 - a = \alpha_3 \alpha_1 x_1 + \alpha_3 \alpha_2 x_2 + \alpha_3^2 x_3 + \alpha_3 a, \quad -\alpha_1 x_3 + x_2 - a - \alpha_3 x_1 = \alpha_2 a + \alpha_2 \alpha_3 x_3 + \alpha_2^2 x_2 + \alpha_2 \alpha_1 x_1.
\]

Denote \(I_c := -c + x + c:\)

\[
-\alpha_2 x_1 - \alpha_1 x_2 + x_3 - a = \alpha_3 \alpha_1 x_1 + \alpha_3 \alpha_2 x_2 + \alpha_3^2 x_3 + \alpha_3 a, \quad -\alpha_1 x_3 + x_2 + I_a(-\alpha_3 x_1) - a = I_{(-\alpha_2 a)} \alpha_3 \alpha_3 x_3 + I_{(-\alpha_2 a)} \alpha_2 x_2 + I_{(-\alpha_2 a)} \alpha_2 \alpha_1 x_1 + \alpha_2 a.
\]

The left and the right sides of these equalities are 0-canonical decompositions of the same operations. By Theorem 1. their coefficients and free members are equal:

\[
-\alpha_2 = \alpha_3 \alpha_1, \quad -\alpha_1 = \alpha_3 \alpha_2, \quad \iota = \alpha_3^2, \quad -a = \alpha_3 a, \quad -\alpha_1 = I_{(-\alpha_2 a)} \alpha_2 \alpha_3, \quad \iota = I_{(-\alpha_2 a)} \alpha_2, \quad I_a(-\alpha_3) = I_{(-\alpha_2 a)} \alpha_2 \alpha_1, \quad -a = \alpha_2 a.
\]

Denote \(\alpha := \alpha_2, \beta := \alpha_3: \)

\[
\alpha = -\beta \alpha_1, \quad \alpha_1 = -\beta \alpha, \quad \iota = \beta^2, \quad \beta a = -a, \quad \alpha_1 = -I_a \alpha \beta, \quad \alpha^2 = I^{-1}_a, \quad -\beta = \alpha \alpha_1, \quad \alpha a = -a.
\]

\[\square\]
6. The symmetry group contains $C_4$.

Since $S_4 = C_4 \sqcup (13)C_4 \sqcup (23)C_4 \sqcup (14)C_4 \sqcup (24)C_4 \sqcup (13)(24)C_4$, then all different parastrophes of an invertible operation $f$ with $\Psi(f) = C_4$ are $f$, $(13)f$, $(23)f$, $(14)f$, $(24)f$, $(13)(24)f$.

**Proposition 7.** A ternary quasigroup $(Q, f)$ belongs to $\Psi(C_4)$ if and only if it satisfies the identities

$$f(y, x, z) = f(x, y, z), \quad f(x, y, f(x, y, z)) = z.$$  \hspace{1cm} (26)

**Proof.** Since $(12)$ and $(34)$ generate the subgroup $C_4$ of the symmetric group $S_4$, then $C_4$ is a subgroup of the symmetry group of a ternary quasigroup $(Q, f)$ if and only if the identity

$$(12)f = f, \quad (34)f = f$$

holds in $(Q, f)$. These equalities could be written as identities in the quasigroup $(Q, f)$:

$$(12)f(x_1, x_2, x_3) = f(x_1, x_2, x_3), \quad (34)f(x_1, x_2, x_3) = f(x_1, x_2, x_3).$$ \hspace{1cm} (27)

Using the definition of parastrophes, we have

$$f(x_2, x_1, x_3) = f(x_1, x_2, x_3), \quad f(x_1, x_2, f(x_1, x_2, x_3)) = x_3.$$\hspace{1cm} (28)

As a result, we obtain the identities (26). \hfill \Box

**Theorem 5.** A ternary group isotope $(Q, f)$ belongs to $\Psi(C_4)$ if and only if there exists an abelian group $(Q, +, 0)$, its permutation $\alpha$ and an element $a \in Q$ such that $\alpha 0 = 0$ and

$$f(x, y, z) = \alpha x + \alpha y - z + a.$$ \hspace{1cm} (29)

**Proof.** Let $(Q, f)$ be a ternary group isotope and (1) be its 0-decomposition. As it was shown above, the quasigroup $(Q, f)$ belongs to the variety $\Psi(C_4)$ if and only if (26) holds. Lemma 1. implies that these identities can be written as

$$\alpha_1 x_2 + \alpha_2 x_1 + \alpha_3 x_3 + a = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a,$$

$$\alpha_3^{-1}(-\alpha_2 x_2 - \alpha_1 x_1 + x_3 - a) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a.$$

In obedience to Lemma 2. and Theorem 1., the first identity means that the group $(Q; +)$ is commutative and $\alpha_1 = \alpha_2 := \alpha$. Therefore, in accordance with Lemma 3., the second identity means that $\beta := \alpha_3$ is an automorphism of $(Q; +)$,

$$-\alpha_3^{-1}a = a.$$ i.e., $\alpha_3 = -\iota$. \hfill \Box

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7. The symmetry group contains $S_2$.

Since
\[ S_4 = \{ S_2 \sqcup (13)S_2 \sqcup (23)S_2 \sqcup (14)S_2 \sqcup (24)S_2 \sqcup (34)S_2 \sqcup (134)S_2 \sqcup (143)S_2 \sqcup (123)S_2 \sqcup (124)S_2 \sqcup (1324)S_2 \sqcup (1423)S_4 \}, \]
then all different parastrophes of an invertible operation $f$ with $\Psi_0(f) = S_2$ are $f$, $(13)f$, $(23)f$, $(14)f$, $(24)f$, $(34)f$, $(134)f$, $(143)f$, $(234)f$, $(243)f$, $(1324)f$, $(1423)f$.

**Proposition 8.** A ternary quasigroup $(Q, f)$ belongs to $\Psi(S_2)$ if and only if it satisfies the identity
\[ f(y, x, z) = f(x, y, z). \] (29)

**Proof.** Evidently. \hfill \Box

**Theorem 6.** A ternary group isotope $(Q, f)$ belongs to $\Psi(S_2)$ if and only if there exists a group $(Q, +, 0)$, its permutation $\alpha$ and an element $a \in Q$ such that $\alpha 0 = 0$ and
\[ f(x, y, z) = \alpha x + \alpha y + \beta z + a. \] (30)

**Proof.** Let $(Q, f)$ be a ternary group isotope and (1) be its 0-decomposition. By Theorem 1., the identity means that the group $(Q; +)$ is commutative and $\alpha_1 = \alpha_2 := \alpha$. Let $\beta := \alpha_3$ be an automorphism of $(Q; +)$. \hfill \Box

8. Conclusion

A class of quasigroups having the same non-trivial parastrophic symmetry group is a variety. There are ten trusses of these varieties. Ten pairwise non-parastrophic varieties are selected for our research. The respective defining systems of identities in propositions 1, 2, 3, 5, 7 in [14] and propositions 1., 3., 5., 7., 8. in this article are found. In each of these varieties, a group isotope of sub-varieties is described.

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**References**


КЛАССИФІКАЦІЯ ТЕРНАРНИХ КВАЗІГРУП ЗА ЇХ ПАРАСТРОФНИМИ ГРУПАМИ СИММЕТРІЙ, ІІ

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РЕЗЮМЕ
Метою цієї та попередньої статті [14] є класифікація тернарних квазігруп за групами їх парастрофної симетрії. Оскільки кожна така група є підгрупою симетричної групи степені 4, тобто \( S_4 \), а парастрофні квазігрупи мають спряжені групи парастрофної симетрії, то в цих працях аналізуються квазігрупи, групи парастрофних симетрій яких попарно неспряжені в \( S_4 \). А саме, знаходяться тотожності, які описують клас квазігруп, група парастрофних симетрій яких містить дану підгрупу групи \( S_4 \); наводиться список різних парастрофів та знаходяться канонічні розклади операцій групових ізотопів, які містяться в даному многовиді.

Key words: тернарна квазігрупа, многовид групи, парастрофні квазігрупи, парастрофна симетрія, парастрофні многовиди, парастрофні групи симетрії