УДК 512.56(2-8)

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ON PARASTROPHICALLY PRIMARY NON-EQUIVALENCE OF GENERALIZED TERNARY QUASIGROUP FUNCTIONAL EQUATIONS OF THE TYPE $(5;3;0;0)$

Two functional equations are called parastrophically primarily equivalent if one can be obtained from the other in a finite number of the following steps: 1) replacing the equation sides; 2) renaming the functional variables; 3) renaming the individual variables; 4) applying the primary identities ([11]). This article is a continuation of the investigation of quasigroup functional equations on ternary quasigroups using the classification method up to parastrophically primary equivalency. In [8] it is proved that there are no more than 36 such equations of the length three, i.e. having three functional variables. In [15] it is established that among them there are exactly 4 functional equations of the type $(2,2,2,2)$ (each individual variable has two appearances). In this article, it is shown that the equations of the type (5;3;0;0) are not parastrophically primarily equivalent to the other ternary equations with three functional variables. All linear solutions of these equations are found. It is proved that the equation $F_1(F_2(y,y,y),x,x) = F_3(x,x,x)$ is not equivalent to the other functional equations of the length three.

Key words: *ternary quasigroups, functional equation, individual type, parastrophically primary equivalence.*

Introduction

If two functional equations are parastrophically primarily equivalent, then there exists a simple dependence between their sets of all solutions. That is why it is efficient to classify functional equations up to parastrophically primary equivalency. There are a lot of articles devoted to the classification of binary functional equations, for example [3, 4, 5, 6, 7, 9, 11, 17, 19].

This article is devoted to classification of ternary functional equations up to parastrophically primary equivalency and it is a continuation of the works [8], [13], [15]. Generalized ternary functional equations of the length one and two are classified in [13]. In [8] it is proved that all functional equations of the length three are parastrophically primarily equivalent to the given list of 36 functional equations. Full classification of generalized quadratic quasigroup functional equations of the length three is given in [15] and their solution sets are found. Because unlike other equations, they have four different individual variables, such equations are not parastrophically primarily equivalent to other functional equations of the length three.

Other ternary generalized functional equations of the length three, namely equations having two different individual variables are under consideration in the current article. It is proved that the equations are not parastrophically primarily equivalent to the other functional equations of the length three. Their linear solution sets are found. In addition, it is established that the equation $F_1(F_2(y,y,y),x,x) = F_3(x,x,x)$ is not equivalent to the other functional equations of the length three from the given list of 36 functional equations.

1. Preliminaries

A mapping $f:\ Q^3 \to Q$ is called a *ternary invertible function*, if there exist functions ${}^{(14)}f, {}^{(24)}f, {}^{(34)}f$ such that for any $x, y, z \in Q$ the following identities:

$$\begin{aligned} f({}^{(14)}f(x,y,z),y,z) &= x, & {}^{(14)}f(f(x,y,z),y,z) &= x,\\ f(x,{}^{(24)}f(x,y,z),z) &= y, & {}^{(24)}f(x,f(x,y,z),z) &= y,\\ f(x,y,{}^{(34)}f(x,y,z)) &= z, & {}^{(34)}f(x,y,f(x,y,z)) &= z \end{aligned} \tag{1}$$

hold. If an operation f is invertible, then the algebra $(Q; f, {}^{(14)}f, {}^{(24)}f, {}^{(34)}f)$ is called a *ternary quasigroup* [12].

A σ*-parastrophe* of an invertible operation f is called an operation ${}^{\sigma}f$ defined by

$${}^{\sigma}f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} \quad :\Leftrightarrow \quad f(x_1, x_2, x_3) = x_4, \quad \sigma \in S_4,$$

where S_4 denotes the group of all bijections of the set $\{1, 2, 3, 4\}$.

Renaming the individual variables, one can conclude that the relationships are equivalent to

$${}^{\sigma}f(x_1, x_2, x_3) = x_4 \quad :\Leftrightarrow \quad f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}) = x_{4\sigma^{-1}}, \quad \sigma \in S_4. \tag{2}$$

If $4\sigma = 4$, the parastrophe is called *principal* and can be found by

$${}^{\sigma}f(x_1, x_2, x_3) = f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}), \quad \sigma \in S_3. \tag{3}$$

Since for every invertible operation f and for every permutation $\sigma \in S_4$ the relations

$${}^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f \quad \text{and} \quad {}^{\iota}f = f \tag{4}$$

hold, then the symmetric group S_4 defines an action on the set Δ_3 of all ternary invertible operations defined on the same carrier. In particular, the fact implies that the number of different parastrophes of an invertible operation is a factor of 24. More precisely, it is equal to $24/|\mathrm{Ps}(f)|$, where $\mathrm{Ps}(f)$ denotes a stabilizer group of f under this action which is called *parastrophic symmetry group* of the operation f.

An operation f is called:

- *totally symmetric*, if all its parastrophes coincide, i.e. $\mathrm{Ps}(f) = S_{n+1}$;
- *commutative*, if all its principal parastrophes coincide, i.e. $\mathrm{Ps}(f) \supseteq S_n$.

A ternary quasigroup $(Q; f)$ is called a *Steiner quasigroup* [10, 16], if it is totally symmetric and satisfies the identity

$$f(x, x, y) = y. \tag{5}$$

Taking into account [14, Proposition 1], the variety of Steiner quasigroups is defined by (5) and

$$f(x, y, z) = f(y, x, z), \qquad f(z, x, f(x, y, z)) = y. \tag{6}$$

The identities (1) are true not only for all values of individual variables in a carrier Q, but for all values of f in the set Δ_3 of all ternary invertible functions defined on Q. That is why (1) can be considered as hyperidentities over Δ_3.

We research *ternary functional equations* which are universally quantified equalities $T_1 = T_2$ where T_1 and T_2 are terms consisting of individual and ternary functional variables. In addition, these equations are considered on an arbitrary set Q called a carrier and therefore they have neither individual nor functional constants [1, 12]. We analyze only generalized ternary quasigroup functional equations of the length three of the type $(5;3;0;0)$, where the collocation *'ternary quasigroup equation'* means that all functional variables take their values only in the set Δ_3; the word *'generalized'* means that the variables are pairwise different [2, 9, 18]; the collocation *'length of a functional equation'* is the number of functional variables including their repetitions [3, 13]; the notion 'individual type' of a functional equation of n individual variables is the sequence $(a_1, a_2, \dots, a_n)$, where a_i is the number of appearances in the equation of the i-th individual variable placed in lexicographic order and n is the number of all possible different independent individual variables [3]. Note that each individual variable appears at least twice in a quasigroup functional equation, otherwise the equation has a solution only on a one-element carrier.

A sequence of values of functional variables of an equation $T_1 = T_2$ is called its *solution*, if the equation becomes a true proposition after replacing the functional variables with these values. If all values coincide with a ternary operation f defined on Q, then the operation f or the groupoid $(Q; f)$ is called a *solution* of $T_1 = T_2$.

Two functional equations are called *parastrophically primarily equivalent* if one can be obtained from the other by renaming functional or individual variables or applying the hyperidentities (1).

Lemma 1. *[15] Let $\upsilon = \omega$ and $\upsilon' = \omega'$ be generalized ternary functional equations of the length three. If they are parastrophically primarily equivalent, then there exists a bijection τ in S_3 and bijections σ_1, σ_2, σ_3 in S_4 such that for an arbitrary solution (f_1, f_2, f_3) of $\upsilon = \omega$ the sequence*

$$({}^{\sigma_1}f_{1\tau}, {}^{\sigma_2}f_{2\tau}, {}^{\sigma_3}f_{3\tau})$$

is a solution of the equation $\upsilon' = \omega'$.

Corollary 1. *[15] If for every bijection τ in S_3 and bijections σ_1, σ_2, σ_3 in S_4 there exists a solution (f_1, f_2, f_3) of $\upsilon = \omega$ such that $({}^{\sigma_1}f_{1\tau}, {}^{\sigma_2}f_{2\tau}, {}^{\sigma_3}f_{3\tau})$ is not a solution of $\upsilon' = \omega'$, then the functional equations $\upsilon = \omega$ and $\upsilon' = \omega'$ are not parastrophically primarily equivalent.*

Corollary 2. *[15] If a totally symmetric function is a solution of a functional equation but it is not a solution of another functional equation, then the equations are not parastrophically primarily equivalent.*

Lemma 2. *[6] If functional equations have a different number of different independent individual variables, then these functional equations are not parastrophically primarily equivalent.*

Theorem 1. *[8] Each ternary quasigroup functional equation of the length three having two different propositional variables is parastrophically primarily equivalent to at least one of the following equations.*

The equations of the type $(6, 2, 0, 0)$ are:

$$F_1(F_2(x, x, x), y, y) = F_3(x, x, x), \tag{7}$$

$$F_1(F_2(x, x, x), x, x) = F_3(x, y, y), \tag{8}$$

$$F_1(F_2(x, x, x), x, y) = F_3(x, x, y), \tag{9}$$

$$F_1(F_2(x, x, y), x, x) = F_3(x, x, y). \tag{10}$$

The equations of the type $(5,3,0,0)$ *are:*

$$F_1(F_2(y,y,y),x,x) = F_3(x,x,x), \tag{11}$$
$$F_1(F_2(x,x,x),x,y) = F_3(x,y,y), \tag{12}$$
$$F_1(F_2(x,x,x),y,y) = F_3(x,x,y), \tag{13}$$
$$F_1(F_2(x,y,y),x,x) = F_3(x,x,y), \tag{14}$$
$$F_1(F_2(x,x,y),x,y) = F_3(x,x,y). \tag{15}$$

The equations of the type $(4,4,0,0)$ *are:*

$$F_1(F_2(x,x,x),x,y) = F_3(y,y,y), \tag{16}$$
$$F_1(F_2(x,x,x),y,y) = F_3(x,y,y), \tag{17}$$
$$F_1(F_2(x,y,y),x,y) = F_3(x,x,y), \tag{18}$$
$$F_1(F_2(x,x,y),y,y) = F_3(x,x,y). \tag{19}$$

2. Linear solutions of equations of the type $(5;3;0;0)$

To analyze the parastrophically primary equivalency of these equations, we must have some sets of their solutions. For this purpose, we find sets of their solutions whose components are linear over the same arbitrary commutative group.

Let $(Q;+,0)$ be an arbitrary commutative group of three operations defined by the equality:

$$f_i(x,y,z) = \alpha_i x + \beta_i y + \gamma_i z + a_i, \tag{20}$$

where α_i, β_i, γ_i are automorphisms of $(Q;+,0)$, $a_i \in Q$, $i = 1,2,3$.

Theorem 2. *A triplet of operations* (f_1, f_2, f_3) *defined by equalities (20) is a solution of the equation*

1. *(11) if and only if*

$$a_3 = a_1 + \alpha_1 a_2, \quad \gamma_2 = -\alpha_2 - \beta_2, \quad \gamma_3 = \beta_1 + \gamma_1 - \alpha_3 - \beta_3; \tag{21}$$

2. *(12) if and only if*

$$a_3 = \alpha_1 a_2 + a_1, \quad \beta_3 = \gamma_1 - \gamma_3, \quad \alpha_3 = \alpha_1(\alpha_2 + \beta_2 + \gamma_2) + \beta_1; \tag{22}$$

3. *(13) if and only if*

$$a_3 = \alpha_1 a_2 + a_1, \quad \gamma_3 = \beta_1 + \gamma_1, \quad \beta_3 = \alpha_1(\alpha_2 + \beta_2 + \gamma_2) - \gamma_3; \tag{23}$$

4. *(14) if and only if*

$$a_3 = \alpha_1 a_2 + a_1, \quad \gamma_3 = \alpha_1\beta_2 + \alpha_1\gamma_2, \quad \beta_3 = \alpha_1\alpha_2 + \beta_1 + \gamma_1 - \alpha_3; \tag{24}$$

5. *(15) if and only if*

$$a_3 = \alpha_1 a_2 + a_1, \quad \gamma_3 = \alpha_1\gamma_2 + \gamma_1, \quad \beta_3 = \alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1 - \alpha_3. \tag{25}$$

Proof. A triplet (f_1, f_2, f_3) of ternary invertible operations is a solution of the equation (11) if and only if for all x and y

$$f_1(f_2(y, y, y), x, x) = f_3(x, x, x).$$

Taking into account the equalities (20), this identity is equivalent to

$$\alpha_1(\alpha_2 y + \beta_2 y + \gamma_2 y + a_2) + \beta_1 x + \gamma_1 x + a_1 = \alpha_3 x + \beta_3 x + \gamma_3 x + a_3.$$

Since α_1 is an automorphism, we get an equivalent identity

$$(\beta_1 + \gamma_1)x + \alpha_1(\alpha_2 + \beta_2 + \gamma_2)y + \alpha_1 a_2 + a_1 = (\alpha_3 + \beta_3 + \gamma_3)x + a_3.$$

Substituting $x = y = 0$, $x = 0$ and $y = 0$ in turn, we obtain $\alpha_1 a_2 + a_1 = a_3$, $\alpha_1(\alpha_2 + \beta_2 + \gamma_2) = 0$, $\beta_1 + \gamma_1 = \alpha_3 + \beta_3 + \gamma_3$. It is obvious that these equalities are equivalent to (21).

A triplet (f_1, f_2, f_3) is a solution of the equation (12) if and only if

$$f_1(f_2(x, x, x), x, y) = f_3(x, y, y)$$

holds. Considering (20), we have

$$\alpha_1(\alpha_2 x + \beta_2 x + \gamma_2 x + a_2) + \beta_1 x + \gamma_1 y + a_1 = \alpha_3 x + \beta_3 y + \gamma_3 y + a_3.$$

This identity is equivalent to $\alpha_1 a_2 + a_1 = a_3$, $\alpha_1(\alpha_2 + \beta_2 + \gamma_2) = \alpha_3$, $\gamma_1 = \beta_3 + \gamma_3$ that are equivalent to (22).

(f_1, f_2, f_3) is a solution of (13) if and only if $f_1(f_2(x, x, x), y, y) = f_3(x, x, y)$ holds. It means that

$$\alpha_1(\alpha_2 x + \beta_2 x + \gamma_2 x + a_2) + \beta_1 y + \gamma_1 y + a_1 = \alpha_3 x + \beta_3 x + \gamma_3 y + a_3.$$

This equality is equivalent to $\alpha_1 a_2 + a_1 = a_3$, $\beta_1 + \gamma_1 = \gamma_3$, $\alpha_1(\alpha_2 + \beta_2 + \gamma_2) = \alpha_3 + \beta_3$. Therefrom we have (23).

A triplet (f_1, f_2, f_3) is a solution of (14) if and only if for all x, y,

$$f_1(f_2(x, y, y), x, x) = f_3(x, x, y)$$

holds. In particular, if we take into account (20), we get

$$\alpha_1(\alpha_2 x + \beta_2 y + \gamma_2 y + a_2) + \beta_1 x + \gamma_1 x + a_1 = \alpha_3 x + \beta_3 x + \gamma_3 y + a_3.$$

This identity is equivalent to the equalities $\alpha_1 a_2 + a_1 = a_3$, $\alpha_1\alpha_2 + \beta_1 + \gamma_1 = \alpha_3 + \beta_3$, $\alpha_1(\beta_2 + \gamma_2) = \gamma_3$. The obtained equalities are identical to (24).

A triplet (f_1, f_2, f_3) is a solution of (15) if and only if for all x, y,

$$f_1(f_2(x, x, y), x, y) = f_3(x, x, y)$$

holds. Taking into account (20), this identity is

$$\alpha_1(\alpha_2 x + \beta_2 x + \gamma_2 y + a_2) + \beta_1 x + \gamma_1 y + a_1 = \alpha_3 x + \beta_3 x + \gamma_3 y + a_3.$$

$\alpha_1 a_2 + a_1 = a_3$, $\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1 = \alpha_3 + \beta_3$, $\alpha_1\gamma_2 + \gamma_1 = \gamma_3$ are equivalent to (23). □

Linear idempotent solutions. Note that a function defined by

$$f_1(x,y,z)=\alpha x+\beta y+\gamma z+a$$

over a commutative group $(Q;+;0)$ is idempotent, i.e., $f(x,x,x)=x$ if and only if $\alpha+\beta+\gamma=\iota$ and $a=0$. Therefore, the canonical decompositions of operations of a triplet (f_1,f_2,f_3) defined by (20) are the following:

$$f_i(x,y,z)=\alpha_i x+\beta_i y+(\iota-\alpha_i-\beta_i)z, \tag{26}$$

where α_i, β_i, $\iota-\alpha_i-\beta_i$ are automorphisms of the commutative group $(Q;+,0)$, $i=1,2,3$.

Corollary 3. *A triplet of invertible idempotent functions (f_1,f_2,f_3) defined by (26) over a commutative group $(Q;+;0)$ is a solution*

- *of (11) if and only if $|Q|=1$;*
- *of (12) if and only if $\alpha_3=\alpha_1+\beta_1$;*
- *of (13) if and only if $\beta_3=\alpha_1-\alpha_3$;*
- *of (14) if and only if $\beta_3=\alpha_1\alpha_2+\iota-\alpha_1-\alpha_3$;*
- *of (15) if and only if $\beta_3=\alpha_1\alpha_2+\beta_1\beta_2+\beta_1-\alpha_3$.*

Proof. Let (f_1,f_2,f_3) be a triplet of ternary invertible operations defined on a commutative group $(Q;+;0)$ by (26), where α_i, β_i, $\iota-\alpha_i-\beta_i$ are automorphisms of the group.

If (f_1,f_2,f_3) is a solution of (11), then by the item 1 of Theorem 2. $\gamma_2=-\alpha_2-\beta_2$, but $\gamma_2=\iota-\alpha_2-\beta_2=\iota+\gamma_2$. Therefore, $\iota=0$, i.e. $x=a$ for all $x\in Q$ and for some $a\in Q$. It means that Q is a singleton.

Let (f_1,f_2,f_3) be a solution of the functional equation (12). Accordingly, from (22):

$$\begin{aligned}\alpha_3=\alpha_1(\alpha_2+\beta_2+(\iota-\alpha_2-\beta_2))+\beta_1=\alpha_1+\beta_1,\\ (\iota-\alpha_1-\beta_1)-(\iota-\alpha_3-\beta_3)=-\alpha_1-\beta_1+\alpha_3+\beta_3=\\ =-\alpha_1-\beta_1+\alpha_1+\beta_1+\beta_3=\beta_3.\end{aligned}$$

Thus, the equality (22) is equivalent to $\alpha_3=\alpha_1+\beta_1$.

Let (f_1,f_2,f_3) be a solution of (13). From the equalities (23)

$$\beta_3=\alpha_1(\alpha_2+\beta_2+\iota-\alpha_2-\beta_2)-\alpha_3=\alpha_1-\alpha_3.$$

That is why, the second equality from (23) follows from the obtained equality:

$$\begin{aligned}\gamma_3\ &=\iota-\alpha_3-\beta_3=\iota-\alpha_3-(\alpha_1-\alpha_3)=\\ &=\iota-\alpha_1=\beta_1+(\iota-\alpha_1-\beta_1)=\beta_1+\gamma_1.\end{aligned}$$

Therefore, item 3) of this theorem is equivalent to item 3) of theorem 2..

Let (f_1,f_2,f_3) be a solution of the functional equation (14). It is equivalent to the equalities (24). Consequently,

$$\beta_3=\alpha_1\alpha_2+\beta_1+(\iota-\alpha_1-\beta_1)-\alpha_3=\alpha_1\alpha_2+\iota-\alpha_1-\alpha_3.$$

The second equality from (24) is equivalent to the obtained one. Indeed,

$$\begin{aligned}\gamma_3 &= \iota - \alpha_3 - \beta_3 = \iota - \alpha_3 - \alpha_1\alpha_2 - \iota + \alpha_1 + \alpha_3 = \alpha_1 - \alpha_1\alpha_2 = \alpha_1(\iota - \alpha_2) = \\ &= \alpha_1(\beta_2 + \iota - \alpha_2 - \beta_2) = \alpha_1\beta_2 + \alpha_1(\iota - \alpha_2 - \beta_2) = \alpha_1\beta_2 + \alpha_1\gamma_2.\end{aligned}$$

Finally, let (f_1, f_2, f_3) be a solution of (15). Therefore, item 5 of theorem 2. holds. Then the third equality follows from the second one:

$$\begin{aligned}\gamma_3 &= \iota - \alpha_3 - \beta_3 = \iota - \alpha_3 - \alpha_1\alpha_2 - \alpha_1\beta_2 - \beta_1 + \alpha_3 = \\ &= \iota - \alpha_1\alpha_2 - \alpha_1\beta_2 - \beta_1 = (\iota - \alpha_1 - \beta_1) + \alpha_1 + \alpha_1(\iota - \alpha_2 - \beta_2 + \iota) = \\ &= \gamma_1 + \alpha_1 + \alpha_1\gamma_2 - \alpha_1 = \gamma_1 + \alpha_1\gamma_2.\end{aligned}$$

□

3. About parastrophically primary equivalency of functional equations of the length three

Every generalized ternary functional equation of the length three is parastrophically primarily equivalent to at least one of the functional equations listed in [8]. The equations having different number of different individual variables are not parastrophically primarily equivalent (Lemma 2.). That is why, each functional equation of the length three with two different individual variables is parastrophically primarily equalent to at least one of the functional equations (7)–(19) (Theorem 1.).

Proposition 1. *Each functional equation of the length three of the type (5;3;0;0) is parastrophically primarily equivalent to none of the rest of the functional equations of the length three.*

Proof. It is easy to verify that an arbitrary Steiner quasigroup is a solution of each of the functional equations (7)–(10) and (16)–(19). But this quasigroup is a solution of none of the equations (11)–(15). According to Corollary 2., the statement of the proposition is true. □

Theorem 3. *The functional equation (11) is parastrophically primarily equivalent to none of the rest of functional equations of the length three.*

Proof. Taking into account Proposition 1., it is enough to prove that the equation (11) is not parastrophically primarily equivalent to the rest of functional equations of the type (5;3;0;0).

Suppose the functional equation (11) is parastrophically primarily equivalent to the functional equation (k), where $k = 12$, 13, 14, 15. Let the solution of the equation (k) consist of three idempotent invertible functions (f_1, f_2, f_3) defined on a set $|Q| > 1$. If the equations (k) and (11) are parastrophically primarily equivalent, then by Lemma 1. there exists a permutation τ in S_3 and permutations σ_1, σ_2, σ_3 in S_4 such that the triplet

$$({}^{\sigma_1}f_{1\tau}, {}^{\sigma_2}f_{2\tau}, {}^{\sigma_3}f_{3\tau})$$

is the solution of the equation (11). But a triplet of idempotent invertible operations is a solution of the equation (11) only if $|Q| = 1$. A contradiction. Therefore, the functional equation (11) is not parastrophically primarily equivalent to the functional equation (k).

Hence, to complete the proof of the theorem, it is sufficient to find examples of triplets of idempotent quasigroups which are solutions of functional equations (12)-(15) on some sets having more than one element. For this purpose, we use Corollary 3..

Example 1. Let a triplet of idempotent invertible operations be defined on the field $\mathbb{Z}_5$ by

$$f_2(x,y,z) = 2x + 2y + 2z,$$
$$f_3(x,y,z) = 2x + 3y + z,$$
$$f_1(x,y,z) = x + y + 4z.$$

Each of the operations is idempotent and invertible. Moreover, the triplet (f_1, f_2, f_3) is a solution of the functional equation (12):

$$2x + 2x + 2x + x + 4y = 2x + 3y + y,$$
$$5x = 0.$$

Hence, the functional equations (11) and (12) are parastrophically primarily nonequivalent.

Example 2. Let a triplet of idempotent invertible operations be defined on the field $\mathbb{Z}_5$ by

$$f_2(x,y,z) = 2x + 2y + 2z,$$
$$f_1(x,y,z) = 2x + 3y + z,$$
$$f_3(x,y,z) = x + y + 4z.$$

The triplet (f_1, f_2, f_3) is a solution of the functional equation (13):

$$2(2x + 2x + 2x) + 3y + y = x + x + 4y,$$
$$10x = 0.$$

Hence, the functional equations (11) and (13) are parastrophically primarily nonequivalent.

Example 3. Let a triplet of idempotent invertible operations be defined on the field $\mathbb{Z}_5$ by

$$f_1(x,y,z) = x + 3y + 3z,$$
$$f_2(x,y,z) = x + 2y + 2z,$$
$$f_3(x,y,z) = x + y + 4z.$$

The triplet (f_1, f_2, f_3) is a solution of the functional equation (13):

$$x + 2y + 2y + 3x + 3x = x + x + 4y,$$
$$5x = 0.$$

Consequently, the functional equations (11) and (14) are parastrophically primarily nonequivalent.

Example 4. A triplet (f_1, f_2, f_2) of idempotent invertible operations defined on $\mathbb{Z}_5$ the ring modulo 5 by

$$f_1(x,y,z) = x + 4y + 2z,$$
$$f_2(x,y,z) = x + 2y + 2z,$$
$$f_3(x,y,z) = x + y + 4z,$$

is a solution of the functional equation (15):

$$(x + 2x + 2y) + 4x + 2y = x + x + 4y, \quad \text{i.e.} \quad \text{i.e.} 5x = 0.$$

Thence, the equations (11) and (15) are not parastrophically primarily equivalent.

Thus, the theorem has been proved. □

Conclusion

The study of classification of ternary quasigroup functional equations up to parastrophically primary equivalence is continued in this article. It is proved that the functional equations having three functional variables (i.e. of the length three) and two individual variables are not parastrophically primarily equivalent to the other functional equations of the length three. Besides, all linear solutions of these equations are found. It is also established that the functional equation (11) is not parastrophically primarily equivalent to the equations (12)–(15).

Acknowledgment

The author would like to express her gratitude to the supervisor Prof. Fedir Sokhatsky for his useful comments and remarks. Furthermore, the author would like to thank the reviewer of English Vira Obshanska who has willingly shared her precious time.

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ПРО ПАРАСТРОФНО-ПЕРВИННУ НЕРІВНОСИЛЬНІСТЬ УЗАГАЛЬНЕНИХ ТЕРНАРНИХ КВАЗІГРУПОВИХ ФУНКЦІЙНИХ РІВНЯНЬ ТИПУ $(5;3;0;0)$

РЕЗЮМЕ

Два функційних рівняння називаються парастрофно-первинно-рівносильними, якщо одне рівняння може бути отримане з іншого за скінченну кількість застосувань наступних кроків: 1) заміни сторін рівняння; 2) перейменування функційних змінних; 3) перейменування предметних змінних; 4) застосування первинних тотожностей ([11]). Ця стаття є продовженням дослідження квазігрупових функційних рівнянь на тернарних квазігрупах з використанням методу класифікації з точністю до парастрофно-первинної рівносильності. В [8] доведено, що існує не більше 36 таких рівнянь довжини три, тобто таких які мають три функційні змінні. У [15] встановлено, що серед них є точно 4 функційних рівняння типу $(2,2,2,2)$ (кожна предметна змінна має дві появи). У цій статті встановлено, що рівняння типу $(5;3;0;0)$ парастрофно-первинно-нерівносильні до решти тернарних рівнянь з трьома функційними змінними. Знайдено всі лінійні розв'язки цих рівнянь і доведено, нерівносильність між рівнянням $F_1(F_2(y,y,y),x,x) = F_3(x,x,x)$ та іншими функційними рівняннями довжини три.

Ключові слова: *тернарна квазігрупа, функційне рівняння, предметний тип, парастрофно-первинна рівносильність.*

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О ПАРАСТРОФНО-ПЕРВИЧНОЙ НЕЕКВИВАЛЕНТНОСТИ ОБОБЩЕННЫХ ТЕРНАРНЫХ КВАЗИГРУППОВЫХ ФУНКЦИОНАЛЬНЫХ УРАВНЕНИЙ ТИПА $(5;3;0;0)$

РЕЗЮМЕ

Два функциональных уравнения называются парастрофно-первично-эквивалентными, если одно уравнение может быть получено с другого за конечное число применений действий: 1) замены сторон уравнения; 2) переименования функциональных переменных; 3) переименования предметных переменных; 4) применения первичных тождеств ([11]). Эта статья является продолжением исследования квазигрупповых функциональных уравнений на тернарных квазигруппах с использованием метода классификации с точностью до парастрофно-первичной эквивалентности. В [8] доказано, что существует не более 36 таких уравнений длины три, то есть таких которые имеют три функциональных переменных. В [15] установлено, что среди них есть точно 4 функциональных уравнения типа $(2,2,2,2)$ (каждая предметная переменная имеет два появления). В этой статье также доказано, что уравнения типа $(5,3,0,0)$ являються парастрофно-первично-неэквивалентны к остальным тернарным уравнениям с тремя функциональными переменными. Найдены все линейные решения этих уравнений и доказана неэквивалентность между уравнением $F_1(F_2(y,y,y),x,x) = F_3(x,x,x)$ и другими функциональными уравнениями длины три.

Ключевые слова: *тернарная квазигруппа, функциональное уравнение, предметный тип, парастрофно-первичная эквивалентность.*