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CLASSIFICATION OF TERNARY QUASIGROUPS ACCORDING TO THEIR PARASTROPHIC SYMMETRY GROUPS, I

The operation f of a ternary quasigroup $(Q;f)$ is called an invertible function. Inverses of f, inverses of inverses and so on, all these operations are called its parastrophes. Each parastrophe ${}^{\sigma}f$ is invertible and uniquely defined by a permutation σ of the set $\{1,2,3,4\}$. Moreover, the symmetric group S_4 acts on the set Δ_3 of all ternary invertible operations which are defined on an arbitrary fixed set. The stabilizer of an operation f is the subgroup $\mathrm{Ps}(f)$ of S_4 and is a set of all σ such that the σ-parastrophe of f coincides with f, i.e., it is the group of all parastrophic symmetries of f. Therefore, the orbit of f is the set of all different parastrophes of f. All quasigroups whose parastrophic symmetry groups include a given group $H \leqslant S_4$ and form a variety $\mathfrak{P}(H)$. The class ${}^{\sigma}\mathfrak{A}$ of all σ-parastrophes of quasigroups from a variety $\mathfrak{A}$ is also a variety. Varieties $\mathfrak{P}(H)$ and $\mathfrak{P}(G)$ are parastrophic if and only if the groups H and G are conjugated. We fix a list of all pairwise unconjugated subgroups of S_4, find a system of axioms of varieties $\mathfrak{P}(H)$ for some groups H from the list and describe subvarieties of group isotopes.

Key words: *ternary quasigroup, variety of a quasigroup, parastrophic quasigroups, parastrophic symmetry, parastrophic varieties, parastrophic symmetry groups.*

Introduction

Quasigroups as algebraic structures are very suitable for construction of cryptographic primitives. There are several classifications of quasigroups. Most of them are made for binary quasigroups.

In [15], the authors considered ternary quasigroups of order 4. To obtain some suitable classification useful for designing cryptographic primitives, the authors investigated the structure of ternary quasigroups. Using some known classifications of binary quasigroups the authors gave a classification of ternary quasigroups of order 4.

In [16], the authors defined ternary quasigroup transformations for encryption and decryption and showed that these transformations are applicable in cryptography for cryptosystems based on quasigroups.

In [18], the notion of using sets with ternary operations to define knot invariants was considered, with colorings of regions in the planar complement of a knot or link diagram by elements of structures known as ternary quasigroups. These structures can be seen as an abstraction of the Dehn presentation of the knot group analogous to the way quandles abstract the Wirtinger presentation.

Recently, ternary quasigroup operations known as Niebrzydowski tribrackets have been studied with additional generalizations to the cases of virtual knots in [17].

In [19], the authors showed a degenerate subcomplex suitable for ternary quasigroups satisfying some axioms, and corresponding to the first Reidemeister move.

The operation in a ternary quasigroup is invertible, i.e., it has three inverse operations: one for each variable. Each inverse operation is also invertible and has three inverses as well and so on. All such inverses are called parastrophes of the given operation and each of the parastrophes is uniquely defined by a permutation of the set $\{1,2,3,4\}$. Moreover, the symmetric group S_4 defines an action on the set of all invertible operations defined on a carrier Q. The stabilizer of an operation is called a parastrophic symmetry group of this operation and it is subgroup of S_4, and the orbit of an operation is a set of all different parastrophes of this operation and is called its truss. If two operations are parastrophic, then their parastrophic symmetry groups are conjugate and their orbits coincide.

A class of quasigroups is called a σ-parastrophe of some other class if it consists only of quasigroups being σ-parastrophic to the quasigroups from the given class. S_4 acts on each set of pairwise parastrophic classes.

In this article, it is proved that a class of quasigroups whose parastrophic symmetry groups contain the given subgroup of the group S_4 forms a variety. Sets of identities which define such varieties are found for all unconjugated subgroups A_3, D_8, S_3, A_4, S_4 of the symmetric group S_4. The necessary and sufficient conditions for a ternary group isotope to belong to each of these varieties are described.

2. Preliminaries

If not specifically stated, all operations are defined on an arbitrary fixed set Q called a *carrier*. In the article, the notions 'n-ary operation' and 'n-ary function' are synonyms and are defined as a mapping from Q^n to Q. The set of all ternary operations defined on Q is denoted by $\mathcal{O}_3$. A ternary operation f is called *invertible* if it is invertible in each of the monoids $(\mathcal{O}_3; \underset{i}{\oplus}, e_i)$, $i=1,2,3$, where

$$(f \underset{1}{\oplus} f_1)(x_1,x_2,x_3) := f(f_1(x_1,x_2,x_3),x_2,x_3),$$
$$(f \underset{2}{\oplus} f_1)(x_1,x_2,x_3) := f(x_1,f_1(x_1,x_2,x_3),x_3),$$
$$(f \underset{3}{\oplus} f_1)(x_1,x_2,x_3) := f(x_1,x_2,f_1(x_1,x_2,x_3)),$$
$$e_i(x_1,x_2,x_3) := x_i,\ i=1,2,3.$$

The operation f is *main* in relation to its inverses ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$ respectively in $(\mathcal{O}_3; \underset{1}{\oplus}, e_1)$, $(\mathcal{O}_3; \underset{2}{\oplus}, e_2)$, $(\mathcal{O}_3; \underset{3}{\oplus}, e_3)$. They are called *left*, *middle* and *right divisions* of f. In other words, the operation f is invertible if the identities

$$f({}^{(14)}f(x,y,z),y,z)=x, \qquad (1) \qquad\qquad {}^{(14)}f(f(x,y,z),y,z)=x, \qquad (4)$$
$$f(x,{}^{(24)}f(x,y,z),z)=y, \qquad (2) \qquad\qquad {}^{(24)}f(x,f(x,y,z),z)=y, \qquad (5)$$
$$f(x,y,{}^{(34)}f(x,y,z))=z, \qquad (3) \qquad\qquad {}^{(34)}f(x,y,f(x,y,z))=z \qquad (6)$$

hold. In this case, the algebra $(Q; f, {}^{(14)}f, {}^{(24)}f, {}^{(34)}f)$ (in brief, $(Q;f)$) is called a *ternary quasigroup* [1]. It is easy to verify that all divisions of an invertible operation are also invertible and so are their divisions. All of these operations having such connections with the main operation f are called parastrophes of f. Namely, a σ*-parastrophe* of an invertible operation f is called an operation ${}^{\sigma}f$ defined by

$${}^{\sigma}f(x_{1\sigma},x_{2\sigma},x_{3\sigma})=x_{4\sigma} \quad :\Leftrightarrow \quad f(x_1,x_2,x_3)=x_4, \quad \sigma\in S_4,$$

where S_4 denotes the group of all bijections of the set $\{0,1,2,3\}$. Therefore in general, every invertible operation has 24 parastrophes. Some of them can coincide.

Since for every invertible operation f and for every permutation $\sigma \in S_4$ the relations

$$^{\sigma}(^{\tau}f) = {}^{\sigma\tau}f \quad \text{and} \quad {}^{\iota}f = f \tag{7}$$

hold, then the symmetric group S_4 defines an action on the set Δ_3 of all ternary invertible operations defined on the same carrier. In particular, the fact implies that the number of different parastrophes of an invertible operation is a factor of 24. More precisely, it is equal to $24/|\mathrm{Ps}(f)|$, where $\mathrm{Ps}(f)$ denotes a stabilizer group of f under this action which is called *parastrophic symmetry group* of the operation f.

Theorem 1. *Let Δ_3 be the set of all ternary invertible operations defined on a set Q, f belong to Δ_3 and let S_4 be the group of all bijections of the set $\overline{1,4} := \{1,2,3,4\}$, then the following assertions are true:*

1. *parastrophy is an equivalence relation on Δ_3. Each block under the action is a set of all pairwise parastrophic operations and is called a* truss *of f, where f is one of the operations and is denoted by $\mathrm{Tr}(f)$;*
2. *$\mathrm{Ps}(f)$ is a subgroup of S_4;*
3. *parastrophic symmetry groups of parastrophic operations are isomorphic, more precisely they are conjugate: $\mathrm{Ps}(^{\sigma}f) = \sigma\mathrm{Ps}(f)\sigma^{-1}$;*
4. *the number of all different parastrophes of f equals $|\mathrm{Tr}(f)| = 24/|\mathrm{Ps}(f)|$;*
5. *If $\mathrm{Ps}(f) = H$, then $^{\sigma}f$, $\sigma \in K$ are all different parastrophes of f, where K is a set of all representatives from the set S_4/H.*

Symmetric group S_4. Let S_n denote the symmetric group of degree n, i.e., the group of all bijections from the set $\{1,\dots,n\}$ onto itself. Therefore, S_4 has 24 members:

$$\begin{aligned} S_4 &= \{\sigma, (04)\sigma, (14)\sigma, (24)\sigma, (34)\sigma \mid \sigma \in S_3\} = \\ &= \{\sigma, \sigma(04), \sigma(14), \sigma(24), \sigma(34) \mid \sigma \in S_3\}. \end{aligned} \tag{8}$$

The equalities (7) and (8) imply that every parastrophe of an invertible operation f is a principal parastrophe of a division of f and equals a division of a principal parastrophe of f.

The set $\mathrm{Sub}(S_4)$ of all subgroups of the summetric group S_4 has 30 elements. The mapping $H \mapsto \tau H \tau^{-1}$ is an action of S_4 on $\mathrm{Sub}(S_4)$. A full set of representatives of the corresponding orbits on $\mathrm{Sub}(S_4)$ is the following and it contains 11 elements:

$$E := \{\iota\}, \quad S_2 := \{\iota, (12)\}, \quad S_{22} := \{\iota, (12)(34)\}, \quad A_3 := \{\iota, (123), (132)\},$$

$$\mathbb{Z}_4 := \{\iota, (12)(34), (1423), (1324)\}, \quad K_4 := \{\iota, (12)(34), (13)(24), (14)(23)\},$$

$$C_4 := \{\iota, (12), (34), (12)(34)\}, \quad S_3 := \{\iota, (12), (13), (23), (123), (132)\},$$

$$D_8 := \{\iota, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}, \quad S_4,$$

$$\begin{aligned} A_4 := \{\iota, (123), (132), (134)(143), (124), (142), (234), (243), \\ (13)(24), (12)(34), (14)(23)\}. \end{aligned}$$

An operation f is called:

- *symmetric*, if some of its parastrophes coincide, i.e. $\mathrm{Ps}(f) \neq \{\iota\}$;
- *asymmetric*, if all its parastrophes are pairwise different, i.e. $\mathrm{Ps}(f) = \{\iota\}$;
- *totally symmetric*, if all its parastrophes coincide, i.e. $\mathrm{Ps}(f) = S_{n+1}$;
- *commutative*, if all its principal parastrophes coincide, i.e. $\mathrm{Ps}(f) \supseteq S_n$;
- *semisymmetric*, if f has at most two different parastrophes, i.e. $\mathrm{Ps}(f) \supseteq A_{n+1}$.
- *dihedrally symmetric* ($n = 3$), if f has at most three different parastrophes, i.e. $\mathrm{Ps}(f) \supseteq D_8$.

According to Theorem 1., the number of different parastrophes of an arbitrary invertible operation f depending on its parastrophic symmetry group is given in the following table:

$\mathrm{Ps}({}^{\sigma}f)$ is	E	S_2	S_{22}	A_3	$\mathbb{Z}_4$	K_4	C_4	S_3	D_8	A_4	S_4
the number of different parastrophes of f is	24	12	12	8	6	6	6	4	3	2	1

3. Group isotopes

A ternary groupoid $(Q; f)$ is called a *group isotope*, if there exists a group $(G; \cdot)$ and bijections α, β, γ, δ from Q to G such that

$$f(x, y, z) = \delta^{-1}(\alpha x \cdot \beta y \cdot \gamma z)$$

for all x, y, z in Q.

Definition 1. *Let $(Q; f)$ be a ternary group isotope and let $(Q; +, 0)$ be a group, α_1, α_2, α_3 be its bijections with $\alpha_1 0 = \alpha_2 0 = \alpha_3 0 = 0$ and $a \in Q$. If*

$$f(x_1, x_2, x_3) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a, \tag{9}$$

then the tuple $(+, \alpha_1, \alpha_2, \alpha_3, a)$ is called a 0-canonical decomposition *of $(Q; f)$; $(Q; +)$ the* canonical decomposition group*; α_1, α_2, α_3 its* coefficients *and a is called a* free member.

Theorem 2. *[3] An arbitrary element of a ternary group isotope uniquely defines its canonical decomposition.*

Lemma 1. *[3] Let $(Q; f)$ be an arbitrary ternary groupisotope and let (9) be its canonical decomposition. Then its divisions and principal parastrophes are*

$$\begin{aligned}
{}^{(14)}f(x_1, x_2, x_3) &= \alpha_1^{-1}(x_1 - a - \alpha_3 x_3 - \alpha_2 x_2),\\
{}^{(24)}f(x_1, x_2, x_3) &= \alpha_2^{-1}(-\alpha_1 x_1 + x_2 - a - \alpha_3 x_3),\\
{}^{(34)}f(x_1, x_2, x_3) &= \alpha_3^{-1}(-\alpha_2 x_2 - \alpha_1 x_1 + x_3 - a),\\
{}^{\sigma}f(x_1, x_2, x_3) &= \alpha_1 x_{1\sigma^{-1}} + \alpha_2 x_{2\sigma^{-1}} + \alpha_{3\sigma^{-1}} x_3 + a, \quad \sigma \in S_3.
\end{aligned} \tag{10}$$

A pair $(Q; \Omega)$ is called a *quasigroup algebra*, if Q is a set and Ω is a set of invertible operations defined on Q. A variable is called *quadratic* in an equality, if it has exactly two appearances in it. Let ω be a term, then $\omega(x)$ means that x has an appearance in ω.

Lemma 2. *[9, Theorem 3] Let $(Q;\cdot,\Omega)$ be a quasigroup algebra besides $(Q;\cdot)$ is a group isotope. If the algebra satisfies an identity*

$$\omega_1(x)\cdot\omega_2(y)=\omega_3(y)\cdot\omega_4(x),$$

where the variables x and y are quadratic, then an arbitrary group being isotopic to the group isotope $(Q;\cdot)$ is commutative.

Lemma 3. *[9, Theorem 3] Let $(Q;+,\Omega)$ be a quasigroup algebra besides $(Q;+,0)$ is a group, $\alpha\in\Omega$ and $\alpha 0=0$. If the algebra satisfies an identity*

$$\alpha(\omega_1(x)+\omega_2(y))=\omega_3(z)+\omega_4(u),$$

then α is: an automorphism of $(Q,+)$, if $z=x$, $u=y$ and anti-automorphism of $(Q,+)$, if $z=y$, $u=x$.

4. Quasigroupes with a fixed parastrophic symmetry group

Let $\mathfrak{P}_Q(H)$ and $\mathfrak{Ps}_Q(H)$ denote the sets of all ternary invertible operations defined on a set Q whose parastrophic symmetry group respectively contains the group $H\in S_4$ and equals the group H. Let $\mathfrak{P}(H)$ and $\mathfrak{Ps}(H)$ denote the class of all quasigroups whose parastrophic symmetry group includes the group $H\in S_4$ and equals the group H respectively. It is easy to see that

- a ternary quasigroup $(Q;f)$ belongs to the class $\mathfrak{P}(H)$ if and only if ${}^{\sigma}f=f$ for all σ from a set of generators of the group H, therefore, the class of quasigroups $\mathfrak{P}(H)$ is a variety;
- the set $\{\mathfrak{Ps}_Q(H)\mid H \text{ is a subgroup of } S_4\}$ is a partition of the set $\Delta_3(Q)$ of all ternary invertible operations defined on Q.

Lemma 4. *If a non-trivial principal parastrophe of a ternary group isotope f coincides with f, then its canonical decomposition group is commutative.*

5. Ternary totally symmetric group isotopes

$\mathfrak{P}(S_4)$ denotes the variety of totally symmetric ternary quasigroups, i.e. their groups of parastrophic symmetries are S_4. It means that all parastrophes of the quasigroups coincide. Since S_4 is generated by (12) and (1234), then the variety is defined by the identities

$$^{(12)}f(x_1,x_2,x_3)=f(x_1,x_2,x_3),\qquad {}^{(1234)}f(x_1,x_2,x_3)=f(x_1,x_2,x_3). \tag{11}$$

As ${}^{(1234)}f={}^{(123)}({}^{(34)}f)$, the identities is equivalent to

$$f(x_2,x_1,x_3)=f(x_1,x_2,x_3),\qquad {}^{(34)}f(x_3,x_1,x_2)=f(x_1,x_2,x_3).$$

Therefore, the following proposition is true.

Proposition 1. *A ternary quasigroup $(Q;f)$ belongs to the totally symmetric ternary quasigroup variety $\mathfrak{P}(S_4)$ if and only if the following identities are valid*

$$f(x,y,z)=f(y,x,z),\quad f(z,x,f(x,y,z))=y. \tag{12}$$

Let $(Q;f)$ be a totally symmetric ternary group isotope, i.e. its group of parastrophic symmetries is S_4. It means that all its parastrophes coincide. Suppose (9) is its canonical decomposition. In particular, ${}^{(123)}f=f$ holds, i.e.,

$$ {}^{(123)}f(x,y,z)=f(x,y,z) $$

for all x, y, z in Q. Using the definition of a parastrophe, we have

$$ f(y,z,x)=f(x,y,z). $$

Apply its canonical decomposition:

$$ \alpha_1 y+\alpha_2 z+\alpha_3 x+a=\alpha_1 x+\alpha_2 y+\alpha_3 z+a $$

According to Lemma 2., the group $(Q;+)$ is commutative. Replacing the variables with 0, we obtain $\alpha_1=\alpha_2=\alpha_3=:\alpha$. Therefore,

$$ f(x,y,z)=\alpha x+\alpha y+\alpha z+a. $$

The formula (10) implies that the equality

$$ {}^{(14)}f(x,y,z)=\alpha^{-1}(x-a-\alpha z-\alpha y). $$

But ${}^{(14)}f=f$. Applying Lemma 1., we have

$$ \alpha^{-1}(x-a-\alpha z-\alpha y)=\alpha x+\alpha y+\alpha z+a $$

for all x, y, z in Q. Applying Lemma 3., we conclude that α is an automorphism of the group $(Q;+)$. Thus, the ternary quasigroup is central. The formula (9) implies that the canonical decomposition of ${}^{(14)}f$ is

$$ {}^{(14)}f(x,y,z)=\alpha^{-1}x+Jy+Jz+J\alpha^{-1}a, $$

where $Jx:=-x$. Theorem 2. about the uniqueness of the canonical decomposition implies the equality $\alpha^{-1}=\alpha$, $J=\alpha$, $J\alpha^{-1}a=a$. This equalities are equivalent to $\alpha=J$.

Thus, we have proved the necessity of the following theorem.

Theorem 3. *A ternary group isotope $(Q;f)$ is totally symmetric if and only if there exists a commutative group $(Q;+)$ and element a such that*

$$ f(x,y,z)=-x-y-z+a. \tag{13} $$

Proof. Sufficiency. Suppose the quasigroup $(Q;f)$ is defined by (13) over a commutative group $(Q;+)$. The quasigroup is commutative because the group $(Q;+)$ is commutative. Therefore, S_3 is a subgroup of a parastrophic symmetry group of the quasigroup $(Q;f)$. Since the set $S_3\cup\{(14)\}$ generates the group S_4, it is enough to prove that ${}^{(14)}f=f$. Indeed,

$$ f(f(x,y,z),y,z)=-(-x-y-z+a)-y-z-a=x+y+z-a-y-z+a=x. $$

The theorem has been proved. □

6. Ternary semisymmetric group isotopes

$\mathfrak{P}(A_4)$ denotes the variety of semisymmetric ternary quasigroups, i.e. their groups of parastrophic symmetries are A_4. Since A_4 is generated by (123) and (124), then the variety is defined by the identities

$$^{(123)}f(x_1,x_2,x_3)=f(x_1,x_2,x_3),\qquad ^{(124)}f(x_1,x_2,x_3)=f(x_1,x_2,x_3). \tag{14}$$

Because $^{(124)}f={}^{(12)}(^{(24)}f)$, the identities is equivalent to

$$f(x_3,x_1,x_2)=f(x_1,x_2,x_3),\qquad ^{(24)}f(x_2,x_1,x_3)=f(x_1,x_2,x_3).$$

Therefore, the following proposition is true.

Proposition 2. *A ternary quasigroup $(Q;f)$ belongs to the totally symmetric ternary quasigroup variety $\mathfrak{P}(A_4)$ if and only if the following identities are valid*

$$f(z,x,y)=f(x,y,z),\qquad f(y,f(x,y,z),z)=x. \tag{15}$$

Let $(Q;f)$ be a ternary semisymmetric group isotope, i.e. its group of parastrophic symmetries is A_4: $\mathrm{Ps}(f)=A_4$ and let (9) be its 0-canonical decomposition. Since $(123)\in A_4$, then $^{(123)}f=f$. As has been shown above, the canonical decomposition group is commutative and all coefficients (9) coincide:

$$f(x_1,x_2,x_3)=\alpha x_1+\alpha x_2+\alpha x_3+a.$$

According to (10):

$$^{(24)}f(x_1,x_2,x_3)=\alpha_2^{-1}(-\alpha_1x_1+x_2-a-\alpha_3x_3),$$

but for every operation g we have

$$^{(13)}g(x_1,x_2,x_3)=g(x_{1(13)},x_{2(13)},x_{3(13)})=g(x_3,x_2,x_1).$$

In particular for $g:={}^{(24)}f$, we obtain

$$^{(13)(24)}f(x_1,x_2,x_3)={}^{(24)}f(x_3,x_2,x_1)=\alpha^{-1}(-\alpha x_3+x_2-a-\alpha x_1).$$

Since $(13)(24)\in A_4$, then $^{(13)(24)}f=f$ and that is why the identity

$$\alpha^{-1}(-\alpha x_3+x_2-a-\alpha x_1)=\alpha x_1+\alpha x_2+\alpha x_3+a$$

is true. Lemma 3. implies that α is an automorphism of the group $(Q;+)$.

It is well-known that (123), (1234) are generators for A_4. Then $A_4\subseteq\mathrm{Ps}(f)$ if and only if the identities

$$^{(123)}f=f,\qquad ^{(1234)}f=f$$

hold. The second identity is

$$^{(1234)}f(x_1,x_2,x_3)=f(x_1,x_2,x_3).$$

According to the definition of a parastrophe, we have

$$f(x_2,x_3,f(x_1,x_2,x_3))=x_1,$$

i.e.,

$$\alpha x_2 + \alpha x_3 + \alpha(\alpha x_1 + \alpha x_2 + \alpha x_3 + a) + a = x_1.$$

Because α is an automorphism of $(Q;+)$,

$$(\alpha + \alpha^2)x_2 + (\alpha + \alpha^2)x_3 + \alpha^2 x_1 + \alpha a + a = x_1$$

This identity is equivalent to the following equalities: $\alpha + \alpha^2 = 0$, $\alpha^2 = \iota$, $\alpha a + a = 0$. This equalities are equivalent to $\alpha = -\iota$. If the group of parastrophic symmetries contains the alternating group, then the quasigroup is totally symmetric. Thus, we have proved the following theorem.

Theorem 4. *Ternary semisymmetric group isotopes do not exist.*

7. Dihedrally symmetric quasigroups

One of generator sets of the group D_8 is $\{(12),(34),(13)(24)\}$. Therefore, the variety $\mathfrak{P}(D_8)$ is defined by the identities

$$^{(12)}f = f, \quad ^{(34)}f = f, \quad ^{(13)(24)}f = f. \tag{16}$$

Consequently, the following assertion is true.

Proposition 3. *A ternary quasigroup $(Q;f)$ belongs to the variety $\mathfrak{P}(D_8)$ if and only if the following identities are valid*

$$f(x,y,z) = f(y,x,z), \quad f(x,y,f(x,y,z)) = z, \quad f(z,f(x,y,z),x) = y. \tag{17}$$

A different collection of identities which describe the variety $\mathfrak{P}(D_8)$ one can obtain by selecting a different set of generators of the group D_8.

Proof. The first equality from (16) means

$$^{(12)}f(x_1,x_2,x_3) = f(x_1,x_2,x_3).$$

Using the definition of a parastrophe, we obtain

$$f(x_{1(12)},x_{2(12)},x_{3(12)}) = f(x_1,x_2,x_3),$$

i.e. this equality is equivalent to the first one from (16).

Analogically, one can prove that the second identity from (16) is equivalent to the second identity from (17). The third equality from (16) means

$$^{(13)}\left(^{(24)}f\right)(x_1,x_2,x_3) = f(x_1,x_2,x_3) \qquad \text{or} \qquad ^{(24)}f(x_3,x_2,x_1) = f(x_1,x_2,x_3).$$

Using the definition of a (24)-parastrophe, we obtain the third identity from (17). □

Proposition 4. *Let $(Q;f)$ be a ternary dihedrally symmetric quasigroup, then only f, $^{(14)}f$, $^{(24)}f$ are different parastrophes of the operation f.*

Proof. It is easy to verify that $S_4/D_8 = \{D_8,(14)D_8,(24)D_8\}$, then only ι, (14), (24) are representatives from S_4/D_8. According to Theorem 1., just f, $^{(14)}f$, $^{(24)}f$ are different parastrophes of the operation f. □

Theorem 5. *A ternary group isotope* (Q, f) *is dihedral if and only if there exists an abelian group* $(Q, +, 0)$, *its involutive automorphism* α *and an element* $a \in Q$ *such that* $\alpha a = -a$ *and*

$$f(x_1, x_2, x_3) = \alpha x_1 + \alpha x_2 - x_3 + a. \tag{18}$$

Proof. Using (10), the equalities (16) can be written as follows:

$$\begin{aligned} &\alpha_1 x_2 + \alpha_2 x_1 + \alpha_3 x_3 + a = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a, \\ &\alpha_3^{-1}(-\alpha_2 x_2 - \alpha_1 x_1 + x_3 - a) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a, \\ &\alpha_2^{-1}(-\alpha_1 x_3 + x_2 - \alpha_3 x_1 - a) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a. \end{aligned} \tag{19}$$

According to Lemma 2., the first equality implies commutativity of the group $(Q; +)$. By Lemma 3., the second and the third equalities imply that α_3 and α_2 are automorphisms of the group.

Taking into account the uniqueness of 0-canonical decomposition, we obtain (19) that is equivalent to the following relationships:

$$\alpha_2 = \alpha_1, \qquad \alpha_3 = -\iota, \qquad \alpha_2^2 = \iota, \qquad \alpha_2 a = -a.$$

Thus, the relations mean that (18) is true. □

8. Commutative ternary quasigroups

One of generator sets of the group S_3 is $\{(12), (13)\}$. Therefore, the variety $\mathfrak{P}(S_3)$ is defined by the identities

$$^{(12)}f = f, \quad ^{(13)}f = f. \tag{20}$$

Consequently, the following assertion is true.

Proposition 5. *A ternary quasigroup* $(Q; f)$ *is commutative if and only if the following identities are valid*

$$f(x, y, z) = f(y, x, z), \quad f(x, y, z) = f(z, y, x). \tag{21}$$

Another collection of identities which describe the variety $\mathfrak{P}(S_3)$ can be obtained by selecting a different set of generators of the group S_3.

Proof. The first equality from (20) means

$$^{(12)}f(x_1, x_2, x_3) = f(x_1, x_2, x_3).$$

Using the definition of a parastrophe, we obtain

$$f(x_{1(12)}, x_{2(12)}, x_{3(12)}) = f(x_1, x_2, x_3),$$

i.e. it is equivalent to the first equality from (21).

Analogically, one can prove that the second identity from (20) is equivalent to the second identity from (21). □

Proposition 6. *Let* $(Q; f)$ *be a ternary quasigroup. If* $\mathrm{Ps}(f) = S_3$, *then only* f, $^{(14)}f$, $^{(24)}f$, $^{(34)}f$ *are different parastrophes of the operation* f.

Proof. It is easy to verify that $S_4/S_3 = \{S_3, (14)S_3, (24)S_3, (34)S_3\}$, then only ι, (14), (24), (34) are representatives from S_4/S_3. According to Theorem 1., just f, ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$ are different parastrophes of the operation f. □

Theorem 6. *A ternary group isotope (Q, f) belongs to $\mathfrak{P}(S_3)$ if and only if there exists an abelian group $(Q, +, 0)$, its bijection α and an element $a \in Q$ such that $\alpha 0 = 0$*

$$f(x_1, x_2, x_3) = \alpha x_1 + \alpha x_2 + \alpha x_3 + a. \tag{22}$$

Proof. Using (10), the equalities (20) can be written as follows:

$$\begin{aligned} \alpha_1 x_2 + \alpha_2 x_1 + \alpha_3 x_3 + a = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a, \\ \alpha_1 x_3 + \alpha_2 x_2 + \alpha_3 x_1 + a = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a. \end{aligned} \tag{23}$$

According to Lemma 2., the first equality implies commutativity of the group $(Q; +)$.

Taking into account the uniqueness of a canonical decomposition, we conclude that (23) is equivalent to the following relationships:

$$\alpha_2 = \alpha_1, \qquad \alpha_3 = \alpha_1.$$

The relations mean that (22) is true. □

9. The group A_3.

Proposition 7. *A ternary quasigroup $(Q; f)$ belongs to the variety $\mathfrak{P}(A_3)$ if and only if the following identities are valid*

$$f(x, y, z) = f(y, z, x). \tag{24}$$

Proof. The group A_3 is generated by (132). Therefore, the variety $\mathfrak{P}(A_3)$ is defined by the identity

$${}^{(132)}f = f. \tag{25}$$

It means that

$${}^{(132)}f(x_1, x_2, x_3) = f(x_1, x_2, x_3).$$

Using the definition of a parastrophe, we obtain

$$f(x_{1(123)}, x_{2(123)}, x_{3(123)}) = f(x_1, x_2, x_3), \quad \text{i.e.,} \quad f(x_2, x_3, x_1) = f(x_1, x_2, x_3).$$

The identity is equivalent to (24). □

Proposition 8. *Let $(Q; f)$ be a ternary quasigroup. If $\mathrm{Ps}(f) = A_3$, then only f, ${}^{(12)}f$, ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$, ${}^{(124)}f$, ${}^{(134)}f$, ${}^{(142)}f$ are different parastrophes of the operation f.*

Proof. One can verify that

$$S_4/A_3 = \{A_3, (12)A_3, (14)A_3, (24)A_3, (34)A_3, (124)A_3, (134)A_3, (142)A_3\},$$

then just ι, (12), (14), (24), (34), (124), (134), (142) are representatives from S_4/A_3. According to Theorem 1., only f, ${}^{(12)}f$, ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$, ${}^{(124)}f$, ${}^{(134)}f$, ${}^{(142)}f$ are different parastrophes of the operation f. □

Theorem 7. *A ternary group isotope* (Q, f) *belongs to* $\mathfrak{P}(A_3)$ *if and only if there exists an abelian group* $(Q, +, 0)$, *its bijection* α *and an element* $a \in Q$ *such that*

$$f(x_1, x_2, x_3) = \alpha x_1 + \alpha x_2 + \alpha x_3 + a. \tag{26}$$

Proof. Using (10), the equalities (25) can be written as follows:

$$\alpha_1 x_3 + \alpha_2 x_1 + \alpha_3 x_2 + a = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a. \tag{27}$$

Taking into account the uniqueness of canonical decomposition, we obtain (27) which is equivalent to the following relationships:

$$\alpha_1 = \alpha_2, \quad \alpha_2 = \alpha_3 \quad \alpha_3 = \alpha_1.$$

The relations mean that (26) is true. □

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КЛАСИФІКАЦІЯ ТЕРНАРНИХ КВАЗІГРУП ЗА ЇХ ПАРАСТРОФНИМИ ГРУПАМИ СИМЕТРІЙ

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РЕЗЮМЕ

Операція f тернарної квазігрупи $(Q; f)$ є оборотною функцією. Інверсії f, інверсії інверсій і т.і., всі ці операції називаються парастрофами. Кожна парастрофа ${}^{\sigma}f$ є оборотною і однозначно визначається перестановкою σ множини $\{1, 2, 3, 4\}$. Більше того, симетрична група S_4 діє на множині Δ_3 всіх оборотних тернарних операцій, які визначені на довільному фіксованому наборі. Стабілізатор операції f є підгрупа $\mathrm{Ps}(f)$ групи S_4 і є множиною всіх σ, таких що σ-парастроф операції f збігається з f, тобто, $\mathrm{Ps}(f)$ є групою всіх парастрофних симетрій f. Тому, орбітою f є множина всіх різних парастроф f. Всі квазігрупи, група симетрій яких містить дану групу $H \leqslant S_4$, утворюють многовид $\mathfrak{P}(H)$. Клас ${}^{\sigma}\mathfrak{A}$ всіх σ-парастроф квазігруп з многовиду $\mathfrak{A}$ також є многовидом. Многовиди $\mathfrak{P}(H)$ і $\mathfrak{P}(G)$ є парастрофними, тоді і тільки тоді, коли групи H і G спряжені. Зафіксовано список всіх попарно неспряжених підгруп S_4, знайдено систему аксіом для визначення многовидів $\mathfrak{P}(H)$ деяких груп H з цього списку і описано підмноговиди ізотопів груп.

Key words: *тернарна квазігрупа, многовид групи, парастрофні квазігрупи, парастрофна симетрія, парастрофні многовиди, парастрофні групи симетрії*

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КЛАССИФИКАЦИЯ ТЕРНАРНЫХ КВАЗИГРУПП В СООТВЕСТВИИ С ИХ ПАРАСТРОФНЫМИ ГРУППАМИ СИММЕТРИЙ

РЕЗЮМЕ

Операция f тернарной квазигруппы $(Q;f)$ является оборотной функцией. Инверсии f, инверсии инверсий и т.д., все эти операции называются парастрофами. Каждый парастроф ${}^{\sigma}f$ является оборотным и однозначно определяется перестановкой σ множества $\{1,2,3,4\}$. Более того, симметричная группа S_4 действует на множестве Δ_3 всех оборотных тернарных операций, которые определены на произвольном фиксированном наборе. Стабилизатор операции f является подгруппой $\mathrm{Ps}(f)$ группы S_4 и является множеством σ, таких что σ-парастроф операции f совпадает с f, то есть, $\mathrm{Ps}(f)$ является группой всех парастрофных симметрий f. Поэтому, орбитой f является множество всех разных парастроф f. Все квазигруппы, группа симметрий которых содержит данную группу $H \leqslant S_4$, образуют многовид $\mathfrak{P}(H)$. Класс ${}^{\sigma}\mathfrak{A}$ всех σ-парастроф квазигрупп из многовида $\mathfrak{A}$ также многовид. Многовиды $\mathfrak{P}(H)$ и $\mathfrak{P}(G)$ будут парастрофными, тогда и только тогда, когда группы H і G сопряжены. Зафиксирован список всех попарно несопряженных подгрупп S_4, найдена система аксиом для определения многовидов $\mathfrak{P}(H)$ некоторых групп H из этого списка и описаны подмноговиды изотопов групп.

Key words: *тернарная квазигруппа, многовид группы, парастрофные квазигруппы, парастрофная симметрия, парастрофные многовиды, парастрофные группы симметрий*