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A COMMUTATIVE BEZOUT DOMAIN IN WHICH EVERY MAXIMAL IDEAL IS PRINCIAL IS AN ELEMENTARY DIVIZOR RING

We prove that a commutative Bezout domain in which every maximal ideal is principal is an elementary divisor ring and its stable range equals 2.

Keywords: an elementary divisor ring, a stable range of ring, *zip* ring, *mzip* ring.

Following Kaplansky [1] a ring R is said to be an elementary divisor ring if every matrix over R is equivalent to a diagonal matrix. Kaplansky proved that if R is an elementary divisor ring then every finitely presented R-module is direct sum of cyclic modules. In [2] the converse to Kaplansky's theorem for commutative ring is proved.

A ring is said to be a Bezout ring if every its finitely generated ideal is principal. In [1] a ring R is said to be a Hermite if every 1×2 and 2×1 matrix over R is equivalent to a diagonal matrix. Obviously an elementary divisor ring is the Hermite and it's easy to see that the Hermite ring is a Bezout. Examples that implications aren't reversible are constructed by Gillman and Henriksen [3].

Henriksen posed a question: is every commutative Bezout domain an elementary divisor domain? [4]

In this paper an affirmative answer in the case of every maximal ideal is principal is given.

All rings will be commutative and have identity. Let's recall several definition and notation.

It is known that in the problems of diagonalization of matrices successfully used the concept of stable range of the ring. It will be useful for us too so it is worth to remind its definition. A stable range of ring R equals 2 ($st.r.(R) = 2$) if for any $a, b, c \in R$ such that $aR + bR + cR = R$ there exist elements $x, y \in R$ such that $(a + cx)R + (b + cy)R = R$ [5].

The following definitions are also important.

Let R be a commutative ring and S be a set of nonzero divisors (regular element) of R. The R_s is a classical ring of quotients of R and we shall denote it by $Q_U(R)$.

Let A be a subset of the ring R. Then define $Ann\, A = \{r \in R |\ rA = 0\}$. The nilradical of R is denoted by $rad\ R$. If $rad\ R = 0$ then R is called a reduced ring.

Let R be a reduced ring and $min\, R$ be minimal prime spectrum of R. If $x \in R$ define $D(x) = \{P \in min\ R\ |x \notin P\}$. Then the sets *D(x)* form a basis for the Zaritsky topology on $min\ R$. When we say that $min\ R$ is compact it means that it is compact in this topology.

Following Faith [6] a ring R is *zip* if:

1) I is an ideal of R ;
2) if $Ann\ I = \{0\}$ then $Ann\ I_1 = \{0\}$ for some finitely generated ideal I_1 such that $I_1 \subseteq I$.

We introduce the concept of *mzip* ring R as a ring with the property: if M is a maximal ideal of R and $Ann\ M = \{0\}$ then $Ann\ M_1 = \{0\}$ for a finitely generated ideal M_1 such that $M_1 \subseteq M$. Clearly, every *zip* ring is a *mzip* ring and if every maximal ideal of ring is finitely generated, then ring is a *mzip* ring.

An ideal I of ring R is dense if it's an annihilator is zero. Thus I is a dense if and only if it is faithfull R-module. If every dense maximal ideal of R contains a regular element then R is evidently *mzip*.

Actually every dense maximal ideal of R contains a regular element if for every maximal ideal of the classical quotient ring $Q_U(R)$ is an annulet. The ring with property that maximal ideals are annulets is called Kasch ring [6]. A ring R is McCoy if every finitely generated dense ideal contains a regular element. (In [7] this is called "Property A".) If I is an ideal of $Q_U(R)$ and $I = I_0 Q_U(R)$, where I_0 is annulet of R (and in this case $I_0 = I \cap R$), one sees that R is McCoy if $Q_U(R)$ is McCoy. If I is a finitely generated dense ideal of a Bezout ring, then $I = aR$ for $a \in R$, so $a \in I$ and a is a regular element. Then a Bezout ring is a McCoy ring and we obtain the following result.

Proposition 1. Let R be a Bezout ring. If R is a *mzip*, then every dense maximal ideal contains a regular element and $Q_U(R)$ is a Bezout and Kasch ring.

Proposition 2. Let R be a reduced Bezout ring in which every maximal ideal is projective. Then R is *mzip* if and only if every maximal ideal of R is principal.

Proof. If M is a dense maximal ideal of R and R is *mzip*, then exists a principal ideal $M_1 = mR$ such that $Ann\, M_1 = \{0\}$ and $M_1 \subset M$. Obviously, m is a regular element. Since $m \in M$ this is possible by [8], then M is a principal ideal.

The proposition is proved.

Let's prove the main results of this paper.

Theorem 1. Let R be a regular *mzip* Bezout ring. Then R is a semi-hereditary ring.

Proof. Let M be a maximal ideal of R. Taking into consideration proposition 1 if M is dense it contains a regular element. Ideal $MQ_U(R)$ of $Q_U(R)$ contains an identity, and hence $MQ_U(R) = Q_U(R)$. Since R is a reduced ring $Ann\, M \neq \{0\}$, therefore $M \cap Ann\, M = \{0\}$ and $M + Ann\, M = R$. Thus $1 = e + n$, where $e \in M$, $n \in Ann\, M$. Hence $M = eR$, where $e = e^2$.

Consider $Q_U(R)$. According to proven above maximal ideals of $Q_U(R)$ are only those are generated by idempotents. By [9] $Q_U(R)$ is a regular ring, namely a direct sum of fields. Since R is a reduced Bezout ring and the fact that $Q_U(R)$ is a regular ring, by [10] it is a semi-hereditary ring.

Theorem is proved.

As consequently theorem 1 we obtain the following result.

Theorem 2. Let R a reduced *mzip* Bezout ring. Then:

1) $min\, R$ is compact;
2) R is a Hermite ring;
3) $st.r.(R) = 2$.

Proof. By theorem 1 R is a semi-hereditary Bezout ring, then $min\, R$ is compact [10]. By [2] R is a Hermite ring and $st.r.(R) = 2$ [11].

Theorem is proved.

In case of elementary divisor ring we get the following result.

Theorem 3. Let R be a Bezout domain and for every non-zero element $a \in R$ the factor-ring $R/rad\,(aR)$ is *mzip*. Then R is an elementary divisor ring.

Proof. Since Bezout domain R is a Hermite ring [12], hence it is sufficient to prove that for all $a, b, c \in R$ such that $aR + bR + cR = R$ there exist $p, q \in R$ such that $paR + (pb + qc)R = R$ [13].

We put $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$.

By using the same terminology as in [2], let M be R-module named by A. It's easy to check that M is a $R/(ac)R$-module. Considering the constraints imposed on R, we have $\bar{R} = R/rad(ac)R$ is reduced *mzip* Bezout ring and by theorem 1, $\bar{R}$ is semi-hereditary ring. Thus $\bar{M} = M/rad(acR)M$ is named by $\bar{A} = \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{b} & \bar{c} \end{pmatrix}$.

Since $\bar{R}$ is Hermite ring and $\bar{a} \cdot \bar{c} = o$ we show as in the proof of [14] that there exist two invertible matrices P and Q and a diagonal matrix D with entries in $\bar{R}$ such that $P\bar{A}Q = D$. We put $D = \begin{pmatrix} \bar{k} & 0 \\ 0 & \bar{e} \end{pmatrix}$.

By [2] we may assume that $\bar{k}$ divides $\bar{e}$. The equality $P^{-1}DQ^{-1} = \bar{A}$ implies that $\bar{a}, \bar{b}, \bar{c} \in \bar{R}\bar{k}$. It follows that $\bar{s}$ is a unit. Hence $\bar{M}$ is a cyclic $\bar{R}$-module.

By Nakayama's lemma it follows that M is cyclic over $R/(ac)R$. Hence M is cyclic over R too. By [2] it follows the existence of elements $p, q \in R$ such that $(pa)R + (bp + cq)R = R$. Hence the matrix A has the canonical diagonal reductions.

Theorem is proved.

Theorem 4. Let R be a Bezout ring in which every maximal ideal is principal. Then $st.r.(R) = 2$.

Proof. Since $st.r.(R) = st.r.(R/rad\, R)$ then we can assume that R is a reduced Bezout ring. Furthermore, since every maximal ideal of $R/rad\, R$ is also principal, then R is a reduced *mzip* Bezout ring.

The previous theorem completes the proof.

Theorem is proved.

Consequently theorems 1, 3 and 4 we obtain the following result.

Theorem 5. A commutative Bezout domain in which every maximal ideal is principal is an elementary divisor ring.

REFERENCES:

1. Kaplansky I. Elementary divisors and modules / I. Kaplansky // Trans. Amer. Math. Soc. – 1949. – V 66. – P. 464–491.
2. Larsen M. Elementary divisor rings and finitely presented modules / M. Larsen, W. Lewis // Trans. Amer. Math. Soc. – 1974. – V. 187. – P. 231–248.
3. Gillman L. Rings of continuous functions in which every finitely generated ideal is principal / L. Gillman, M. Henriksen // Trans. Amer. Math. Soc. – 1956. – V. 82. – P. 366–394.

4. Henriksen M. Some remarks about elementary divisor rings / M. Henriksen // Michigan Math. J. – 1955/56. – V. 3. – P. 159–163.
5. Vasserstein L.N. The stable rank of rings and dimensionality of topological spaces / L.N. Vasserstein // Functional Anal. Appl. – 1971. – V 5. – P. 102–110.
6. Faith C. Rings with zero intersection property on annihilators: Zip rings / Faith C. // Publications Math. – 1989. – V 33. – P. 329–332.
7. Huckaba J. Commutative rings with zero divisors / J. Huckaba // Monograph in Pure and Applied. Math. Marcel Dekker. – Basel and New York. – 1988. – P. 331–335.
8. Bourbaki N. Linear algebra / N. Bourabaki // Algebra. – Hermann, Paris. – 1961. – V. 1. – P. 191-425.
9. Satyanarayana M. Rings with primary ideals as maximal ideals / M. Satyanarayana // Math scand. – 1967. – V 20. – P. 52–54.
10. Matlis F. The minimal prime spectrum of a reduced ring / F. Matlis // Illinois J. Math. – 1983. – V. 27, № 3. – P. 353–391.
11. Menal P. On regular rings with stable rage 2 / P. Menal, J. Moncasi // J. Pure Appl. Algebra. – 1982. – V. 24. – P. 25–40.
12. Amitsur S.A. Remarks of principal ideal rings / S.A. Amitsur // Osaka Math. J. – 1963. – V 15. – P. 59–69.
13. Gillman L. Some remarks about elementary divisor rings / L. Gillman, M. Henriksen // Trans. Amer. Math. Soc. – 1956. – V 82. – P. 362–365.
14. Shores T.S. Modules over semi hereditary Bezout rings / T.S. Shores // Proc. Amer. Math. Soc. – 1974. – V. 46. – P. 211–213.

КОМУТАТИВНА ОБЛАСТЬ БЕЗУ, В ЯКІЙ ДОВІЛЬНИЙ МАКСИМАЛЬНИЙ ІДЕАЛ Є ГОЛОВНИМ, Є КІЛЬЦЕМ ЕЛЕМЕНТАРНИХ ДІЛЬНИКІВ

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РЕЗЮМЕ

Показано, що комутативна область Безу, в якій довільний максимальний ідеал є головним, є кільцем елементарних дільників, і стабільний ранг такої області Безу дорівнює 2.

Ключові слова: кільце елементарних дільників, стабільний ранг кільця, *zip* кільце, *mzip* кільце.

КОММУТАТИВНАЯ ОБЛАСТЬ БЕЗУ, В КОТОРОЙ МАКСИМАЛЬНЫЙ ИДЕАЛ ЯВЛЯЕТСЯ ГЛАВНЫМ, ЯВЛЯЯСЬ КОЛЬЦОМ ИДЕАЛЬНЫХ ДЕЛИТЕЛЕЙ

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РЕЗЮМЕ

Показано, что коммутативная область Безу, в которой произвольный максимальный идеал является главным, является кольцом элементарных делителей и стабильный ранг такой области равен 2.

Ключевые слова: кольцо элементарных делителей, стабильный ранг кольца, *zip* кольцо, *mzip* кольцо.