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VARIETY OF SEMISYMMETRY-LIKE MEDIAL QUASIGROUPS AND ITS SUBVARIETIES

In this paper, the identities defining the varieties $\mathcal{S}$, $\mathcal{S}_1$, $\mathcal{S}_2$, which are similar to the variety of semisymmetric group isotopes are described. The conditions of coinciding quasigroups from $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ and semisymmetric isotopes of a commutative group are established. According to symmetry concept, these three varieties and their parastrophic varieties, the quasigroups belonging to all these varieties are described. The relationships among all these varieties are shown.

Keywords: quasigroup, identity, parastrophe, symmetry, qroup isotope, variety, medial, asymmetric, middle symmetric, semisymmetric, totally symmetric.

Introduction

D.C. Murdoch [1] showed that every medial quasigroup, i.e., a quasigroup defined by the equality $xy \cdot uv = xu \cdot yv$, is isotopic to some abelian group. Medial quasigroups are obtained by relabelling the entries and re-arranging the rows and columns of a Latin square of some abelian group. R.H. Bruck gave a general method of constructing medial quasigroups from an abelian group (it is a well-known Toyoda-Bruck theorem [2]). *T*-quasigroups coincide with their center in the class of quasigroups and have the same role as abelian groups in the class of all groups. Every medial quasigroup coincides with its center in class of quasigroups, because it is a *T*-quasigroup. For *T*-quasigroups this fact was shown by G.B. Belyavskaya [3]. Medial quasigroups and related problems were studied by J. Ježek and T. Kepka [4], K.K. Shchukin [5], V.A. Shcherbacov [6], F.M. Sokhatsky [7].

According to A. Sade [8], a groupoid or a quasigroup which satisfies the identity $xy \cdot x = y$ is called semisymmetric. He also established properties and structure of semisymmetric quasigroups. I.M.H. Etherington [9] and A. Sade [8] showed that every semisymmetric groupoid is necessarily a semisymmetric quasigroup. V.V. Iliev [10] studied a construction of the semisymmetric algebras over a commutative ring with unit. F. Radó [11] found the necessary and sufficient conditions for existence of semisymmetric group isotopes of prime order. H. Krainichuk [12] established the criterion of semisymmetry of group isotopes of an arbitrary quasigroup.

F. Sokhatsky [13] proposed a symmetry concept for parastrophes of quasigroup varieties and their quasigroups. This symmetry concept is used for investigation of parastrophes of quasigroup varieties and, in particular, quasigroups and their parastrophes. F. Sokhatsky's symmetry concept generalizes the symmetry known as triality which was investigated by J.D.H. Smith [14]. If a σ-parastrophe coincides with a quasigroup itself, then σ is called a symmetry of the quasigroup. The set of all symmetries of a binary quasigroup forms a group, which is a subgroup of the symmetry group S_3. According to the symmetry group, there are six classes (varieties) of quasigroups: middle symmetric, left symmetric, right symmetric, semisymmetric, totally symmetric and asymmetric (which consists of quasigroups with a unitary symmetry group).

An identity with mutually invers coeficients in its canonical conditions will be called a semisymmetry-like identity. A variety defined by such identity will be called a semisymmetry-like variety. In this article, the following 10 identities

$x \cdot (yu \cdot v) = y \cdot (xu \cdot v),$	(i)	$(x \cdot yu) \cdot v = (x \cdot yv) \cdot u,$	(ii)
$((xy \cdot u) \overset{l}{\cdot} v) \cdot x = vy \cdot u,$	(iii)	$(x \overset{r}{\cdot} yu) \cdot yv = u \cdot xv,$	(iv)
$x \cdot (y \overset{r}{\cdot} (u \cdot vx)) = u \cdot vy,$	(v)	$xy \cdot (uy \overset{l}{\cdot} v) = xv \cdot u,$	(vi)
$x \cdot (y \cdot uv) = (vx \cdot y) \cdot u,$	(vii)	$x \cdot (y \cdot ((u \cdot xv))) \cdot vy = u,$	(viii)
$xy \cdot (((yu \cdot v) \cdot x) \cdot u) = v,$	(ix)	$(xy \cdot u) \cdot (x \cdot vu) = yv$	(x)

are considered.

The identity (i) is semisymmetry-like and defines the variety of semisymmetry-like quasigroups, the identities (ii)-(x) define some its subvarieties and their parastrophes. The listed identities define seven varieties which are distributed into three trusses. Throughout the article, we will use the following notations: $\mathcal{S}$ denotes a variety defined by the identitity (i); $\mathcal{S}_1$ denotes a variety defined by the identitity (vii); $\mathcal{S}_2$ denotes a variety defined by the identitity (x). All identities imply group isotopism that is why they are considered in the class of all group isotopes. A notion of canonical conditions of an identity is introduced to establish connections among the identities and to describe quasigroups belonging to the corresponding varieties.

In this paper, canonical conditions of all listed identities are found, spesificaly, relationships among canonical conditions of parastrophic identities are established (Theorem 5, Lemma 14, Lemma 15). Varieties $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ contain isotopes of an abelian group. They are subvarieties of medial quasigroups, which are similar to the variety of semisymmetric group isopopes. Coefisients of the canonical conditions of the identities defining all these varieties and coefisients of the canonical decompositions of all semisymmetric group isotopes coincide. All quasigroups belonging to $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ and their parastrophic varieties are described. In particular, constructive conditions for building semisymmetric group isotopes are established (Corollaries 19-22). According to the symmetry concept, varieties and corresponding trusses of varieties are characterized (Theorems 6-8).

1. Preliminaries

A *quasigroup* is an algebra $(Q;\cdot,\overset{l}{\cdot},\overset{r}{\cdot})$ satisfying the identities

$$(x\cdot y)\overset{l}{\cdot} y = x, \qquad (x\overset{l}{\cdot} y)\cdot y = x, \qquad x\cdot(x\overset{r}{\cdot} y) = y, \qquad x\overset{r}{\cdot}(x\cdot y) = y. \tag{1}$$

The operation $(\cdot)$ is called *main* and $(\overset{l}{\cdot})$, $(\overset{r}{\cdot})$ are its *left* and *right divisions*. These operations and their dual, which are defined by

$$x\overset{s}{\cdot} y := y\cdot x, \qquad x\overset{sl}{\cdot} y := y\overset{l}{\cdot} x, \qquad x\overset{sr}{\cdot} y := y\overset{r}{\cdot} x$$

are called *parastrophes* of $(\cdot)$ and the defining identities are called *primary*. Last three relationships establish bijection among identities of signature $(\cdot,\overset{l}{\cdot},\overset{r}{\cdot})$ and $(\cdot,\overset{l}{\cdot},\overset{r}{\cdot},\overset{s}{\cdot},\overset{sl}{\cdot},\overset{sr}{\cdot})$. Therefore, throughout the article we consider identities on quasigroups of signature $(\cdot,\overset{l}{\cdot},\overset{r}{\cdot},\overset{s}{\cdot},\overset{sl}{\cdot},\overset{sr}{\cdot})$. All parastrophes of $(\cdot)$ can be defined by

$$x_{1\sigma}\overset{\sigma}{\cdot} x_{2\sigma} = x_{3\sigma} \quad :\Leftrightarrow \quad x_1\cdot x_2 = x_3,$$

where $\sigma\in S_3 := \{\iota, l, r, s, sl, sr\}$, $s := (12)$, $l := (13)$, $r := (23)$. It is easy to verify that

$${}^{\sigma}\begin{pmatrix}\tau\\ \cdot\end{pmatrix} = \begin{pmatrix}\sigma\tau\\ \cdot\end{pmatrix}$$

holds for all $\sigma,\tau\in S_3$.

1.1. Notations and definitions.

Transformation of an identity id to an identity id' using primary identities, external left and external right divisions is called *primary (parastrophic) transformation* [15].

It easy to see that a mapping $(\sigma;(\cdot))\mapsto(\overset{\sigma}{\cdot})$ is an action on the set of all quasigroup operations on Q. A stabilizer $\mathrm{Sym}(\cdot)$ is called a *parastrophic symmetry group* of $(\cdot)$. Thus, the number of different parastrophes of a quasigroup operation $(\cdot)$ depends on its symmetry group $\mathrm{Sym}(\cdot)$. Since $\mathrm{Sym}(\cdot)$ is a subgroup of S_3, then there are six classes of quasigroups. A quasigroup is called

- *asymmetric*, if $\mathrm{Sym}(\cdot)=\{\iota\}$, i.e., all parastrophes are pairwise different;
- *middle symmetric* or *commutative*, if $\mathrm{Sym}(\cdot)\supseteq\{\iota,s\}$, i.e., it satisfies $xy=yx$;
- *left symmetric*, if $\mathrm{Sym}(\cdot)\supseteq\{\iota,r\}$, i.e., it satisfies $x\cdot xy=y$;
- *right symmetric*, if $\mathrm{Sym}(\cdot)\supseteq\{\iota,l\}$, i.e., it satisfies $xy\cdot y=x$;
- *semi-symmetric*, if $\mathrm{Sym}(\cdot)\supseteq A_3$, i.e., it satisfies $x\cdot yx=y$;
- *totally symmetric*, if $\mathrm{Sym}(\cdot)=S_3$, i.e., all parastrophes coincide and in other words $xy=yx$ and $x\cdot xy=y$ hold.

Let P be an arbitrary proposition in a class of quasigroups $\mathcal{A}$. The proposition ${}^{\sigma}P$ is said to be a σ-parastrophe *of* P, if it can be obtained from P by replacing every τ-parastrophe with $\tau\sigma^{-1}$ for every $\tau\in S_3$;

${}^{\sigma}\mathcal{A}$ denotes the class of all σ-parastrophes of quasigroups from $\mathcal{A}$.

Theorem 1. [13] *Let $\mathcal{A}$ be a class of quasigroups, then a proposition P is true in $\mathcal{A}$ if and only if ${}^{\sigma}P$ is true in ${}^{\sigma}\mathcal{A}$.*

Corollary 1. [13] *Let P be true in a class of quasigroups $\mathcal{A}$, then ${}^{\sigma}P$ is true in $\mathcal{A}$ for all $\sigma \in \mathrm{Sym}\mathcal{A}$.*

Corollary 2. [13] *Let P be true in a totally symmetric class $\mathcal{A}$, then ${}^{\sigma}P$ is true in $\mathcal{A}$ for all σ.*

Corollary 3. [13] *Symmetry groups of parastrophic varieties are conjugate, that is*

$$\mathrm{Sym}({}^{\sigma}\mathcal{A}) = \sigma(\mathrm{Sym}\mathcal{A})\sigma^{-1}.$$

Proposition 4. *If $\mathcal{A} \subseteq \mathcal{B}$, then ${}^{\sigma}\mathcal{A} \subseteq {}^{\sigma}\mathcal{B}$ for every $\sigma \in S_3$.*

Definition 1. Transformation of the identity id to the identity ${}^{\sigma}id$ is called a σ*-parastrophic transformations*, if it can be obtained by replacing main operation with some its σ^{-1}-parastrophe.

Two identities are called

- *equivalent*, if they define the same variety;
- σ*-parastrophic*, if one of them can be obtained from the other by a σ-parastrophic transformation (according to Theorem 1, σ-parastrophic identities define σ-parastrophic varieties).

Example 1. Let identity (i) is a proposition P. The proposition ${}^{s}P$, i.e., identity, which is s-parastrophic to (i) can be obtained from (i) by replacing every ι-parastrophe with ιs-parastrophe. Thus, s-parastrophe of the identity (i) has the form

$$x \overset{s}{\cdot} ((y \overset{s}{\cdot} u) \overset{s}{\cdot} v) = y \overset{s}{\cdot} ((x \overset{s}{\cdot} u) \overset{s}{\cdot} v).$$

Using the definition of s-parastrophe, we obtain $(v \cdot uy) \cdot x = (v \cdot ux) \cdot y$. It coincides with (ii).

According to Theorem 1, if proposition P defines a class of quasigroups $\mathcal{A}$, then ${}^{\sigma}P$ defines ${}^{\sigma}\mathcal{A}$. It means, the identity (i) defines variety $\mathcal{S}$, therefore s-parastrophe of (i), i.e., (ii) defines variety ${}^{s}\mathcal{S}$.

1.2. On group isotopes. A groupoid $(Q;\cdot)$ is called an *isotope of a groupoid* $(Q;+)$ iff there exists a triple of bijections (δ,ν,γ), such that the relation $x \cdot y = \gamma(\delta^{-1}x + \nu^{-1}y)$ holds. The triple of bijections is called an *isotopism.* An isotope of a group is termed a *group isotope*.

Definition 2. [16] Let $(Q;\cdot)$ be a group isotope and 0 be an arbitrary element of Q, then the right part of the formula

$$x \cdot y = \alpha x + a + \beta y \tag{2}$$

is called a 0-*canonical decomposition*, if $(Q;+)$ is a group, 0 is its neutral element and $\alpha 0 = \beta 0 = 0$. We will say that 0 defines the *canonical decomposition*, $(Q;+)$ is its *decomposition group*, α and β are its *left* and *right coefficients*, a is its *free member*. A canonical decomposition will be denoted by $(+,0,\alpha,\beta,a)$.

Theorem 2. [16] *An arbitrary element of a group isotope uniquely defines a canonical decomposition of the isotope.*

Recall a variable is *quadratic* in an identity, if it has exactly two appearances in this identity. An identity is called *quadratic*, if all variables are quadratic. An identity is called *gemini* [17], if it is a trivial equality in an arbitrary *TS*-loop.

Theorem 3. [17] *If a quasigroup satisfies a non-gemini identity, then it is isotopic to a group.*

Corollary 5. [18] *If a group isotope $(Q;\cdot)$ satisfies the identity*

$$w_1(x) \cdot w_2(y) = w_3(y) \cdot w_4(x) \tag{3}$$

and the variables x, y are quadratic, then every group being isotopic to $(Q;\cdot)$ is commutative.

It is easy to verify that the following proposition is true.

Proposition 6. *A triple (α,β,γ) of permutations of a set Q is an autotopism of a commutative group $(Q;+)$ if and only if there exists an automorphism θ of $(Q;+)$ and elements $b,c \in Q$ are such that*

$$\alpha = L_{c-b}\theta, \qquad \beta = L_b\theta, \qquad \gamma = L_c\theta.$$

This proposition immediately follows

Corollary 7. *Let $(Q;+,0)$ be an abelian group, $\alpha,\beta_1,\beta_2,\beta_3,\beta_4$ permutations of Q, $\alpha 0 = 0$ and*

$$\alpha(\beta_1 x + \beta_2 y) = \beta_3 x + \beta_4 y,$$

holds for all $x, y \in Q$, *then* α *is an automorphism of the group.*

It is easy to verify that the following lemma holds.

Lemma 8. *Let* $(Q;+)$ *be an abelian group,* α_i, β_i *be its automorphisms and* a_i, b_i *be its elements,* $i = 1, 2, 3, 4$. *Then the equality*

$$\alpha_1 x_1 + a_1 + \alpha_2 x_2 + a_2 + \alpha_3 x_3 + a_3 + \alpha_4 x_4 + a_4 = \beta_1 x_1 + b_1 + \beta_2 x_2 + b_2 + \beta_3 x_3 + b_3 + \beta_4 x_4 + b_4$$

is equivalent to $\alpha_i = \beta_i$, $i = 1, 2, 3, 4$, $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$.

It is easy to prove that parastrophes of an isotope of a commutative group have the following forms

$$\begin{aligned} x \overset{\iota}{\cdot} y &= \alpha x + a + \beta y; & x \overset{s}{\cdot} y &= \beta x + a + \alpha y; \\ x \overset{l}{\cdot} y &= \alpha^{-1}(x - a - \beta y); & x \overset{sl}{\cdot} y &= \alpha^{-1}(-\beta x - a + y); \\ x \overset{r}{\cdot} y &= \beta^{-1}(-\alpha x - a + y); & x \overset{sr}{\cdot} y &= \beta^{-1}(x - a - \alpha y); \end{aligned} \qquad (4)$$

Proposition 9. *A quasigroup is a group if and only if coefficients of its canonical decomposition are identical permutations.*

Systematizing all criteria on symmetry, H.V. Krainichuk [12] gave a classification of group isotopes according to their symmetry groups and formulated the following corollary about classification of isotopes of commutative groups.

Corollary 10. [12] *Let* $(Q;\cdot)$ *be an isotope of a commutative group and* (2) *be its canonical decomposition, then* $(Q;+)$ *is abelian and*

1) $(Q;\cdot)$ *is commutative if and only if* $\alpha = \beta$;
2) $(Q;\cdot)$ *is left symmetric if and only if* $\beta = -\iota$;
3) $(Q;\cdot)$ *is right symmetric if and only if* $\alpha = -\iota$;
4) $(Q;\cdot)$ *is totally symmetric if and only if* $\alpha = \beta = -\iota$;
5) $(Q;\cdot)$ *is semi-symmetric if and only if* α *is an automorphism of* $(Q;+)$, $\beta = \alpha^{-1}$, $\alpha^3 = -\iota$, $\alpha a = -a$;
6) $(Q;\cdot)$ *is asymmetric if and only if* $-\iota \ne \alpha \ne \beta \ne -\iota$ *and at least one of the following conditions is true:* α *is not an automorphism,* $\beta \ne \alpha^{-1}$, $\alpha^3 \ne -\iota$, $\alpha a \ne -a$.

2. Parastrophic identities

In this section, we prove that parastrophic equivalency divides the given identities (i)-(x) into three blocks: (i)-(vi), (vii)-(ix) and (x).

Lemma 11. *The identities* (ii), (iii), (iv), (v), (vi) *are* s-, l-, r-, sl-, sr-*parastrophes of* (i) *respectively.*

Proof. In Example 1, we have shown that s-parastrophe of (i) is (ii).

According to the definition of parastrophic statements, l-parastrophe of (i) is the following identity

$$x \overset{l}{\cdot}((y \overset{l}{\cdot} u) \overset{l}{\cdot} v) = y \overset{l}{\cdot}((x \overset{l}{\cdot} u) \overset{l}{\cdot} v).$$

Replace y with $yv \cdot u$ and x with $xv \cdot u$ and apply the first identity of (1):

$$(xv \cdot u) \overset{l}{\cdot} y = (yv \cdot u) \overset{l}{\cdot} x.$$

According to the first identity from (1) for subterms, we obtain (iii), so l-parastrophe of (i) is (iii).

r-parastrophe of the identity (i) is

$$x \overset{r}{\cdot}((y \overset{r}{\cdot} u) \overset{r}{\cdot} v) = y \overset{r}{\cdot}((x \overset{r}{\cdot} u) \overset{r}{\cdot} v).$$

Apply the third identity of (1) for subterms:

$$x \cdot (y \overset{r}{\cdot}((x \overset{r}{\cdot} u) \overset{r}{\cdot} v)) = (y \overset{r}{\cdot} u) \overset{r}{\cdot} v.$$

Replacing u with xu in the obtained identity and using the third identity from (1), we have

$$x \cdot (y \overset{r}{\cdot}(u \overset{r}{\cdot} v)) = (y \overset{r}{\cdot} xu) \overset{r}{\cdot} v.$$

Substitute $u \cdot yv$ for v, then by the third identity of (1), we obtain

$$xv = (y \overset{r}{\cdot} xu) \overset{r}{\cdot} (u \cdot yv).$$

According to the third identity from (1) for subterms, we receive $(y \overset{r}{\cdot} xu) \cdot xv = u \cdot yv$, that is the identity (iv). It means (iv) is r-parastrophe of (i).

s-parastrophes of (iii) and (iv) give (v) and (vi) respectively, that is sl- and sr- parastrophes of the identity (i) respectively. □

Lemma 12. *Identity* (vii) *and its s-parastrophe coincide. l- and sr-parastrophes, r- and sl-parastrophes of* (vii) *coincide with* (viii) *and* (ix) *respectively.*

Proof. Can be proved similarly to Lemma 11. □

Lemma 13. *All parastrophes of the identity* (x) *coincide with* (x).

Proof. To prove this lemma it is enough to show that s- and l-parastrophes of the identity (x) coincide with (x), since $\{s, l\}$ generates the group S_3. □

3. Canonical conditions of identities

It is easy to see that none of the identities (i)-(x) is gemini, so by Theorem 3, every quasigroup satisfying at least one of the identities is a group isotope. We introduce a notion of canonical conditions, which is a very effective tool for describing subclasses of a group isotope variety and for establishing their properties.

Definition 3. Conditions for components of a canonical decomposition are called *canonical conditions of an identity*, if they hold exclusively for those group isotopes, which satisfy the identity.

Using the given definition, Toyoda-Bruck theorem [3, Theorem 2.10, p. 33] can be reformulated as follows:

Theorem 4. [3] The variety of medial quasigroups is a subvariety of the variety of all group isotopes and is described by the following canonical conditions: commutativity of canonical decomposition group and commuting of its coefficients.

Canonical conditions of the identities (i), (vii), (x) are described in the following theorem:

Theorem 5. *Varieties being defined by the identities* (i), (vii), (x) *are subvarieties of the variety of all medial quasigroups and are described in this variety by the following canonical conditions:*

1) $\beta = \alpha^{-1}$ *for the identity* (i)*;*

2) $\beta = \alpha^{-1}$, $\alpha^6 = \iota$, $\alpha^4 a + \alpha^3 a - \alpha a - a = 0$ *for the identity* (vii)*;*

3) $\beta = \alpha^{-1}$, $\alpha^3 = -\iota$ *for the identity* (x)*,*

where $(+, 0, \alpha, \beta, a)$ *denotes a canonical decomposition of a group isotope.*

Proof. As was mentioned above, every quasigroup satisfying at least one of the identities (i), (vii), (x) is a group isotope. Let (2) be its canonical decomposition.

1) Let $(Q;\cdot)$ be a quasigroup satisfying the identity (i), then it is a group isotope. Since quadratic variables x and y satisfy (3), then by Corollary 5, $(Q;+)$ is a commutative group. Applying (2) to (i), we have:

$$\alpha x + a + \beta(\alpha(\alpha y + a + \beta u) + a + \beta v) = \alpha y + a + \beta(\alpha(\alpha x + a + \beta u) + a + \beta v). \tag{5}$$

Put $u = v = 0$ in (5) then according to Corollary 7, β is an automorphism. Analogically, putting $x = v = 0$, we receive an automorphism $\beta\alpha$. Herefrom, α is an automorphism as well.

Since α and β are automorphisms of a commutative group $(Q;+)$, the identity (5) has the form

$$\alpha x + a + \beta\alpha^2 y + \beta\alpha a + \beta\alpha\beta u + \beta a + \beta^2 v = \alpha y + a + \beta\alpha^2 x + \beta\alpha a + \beta\alpha\beta u + \beta a + \beta^2 v.$$

By Lemma 8, the last equality is tantamount to the conjunction

$$\alpha = \beta\alpha^2, \quad \beta\alpha^2 = \alpha, \quad \beta\alpha\beta = \beta\alpha\beta, \quad \beta^2 = \beta^2, \quad a + \beta\alpha a + \beta a = a + \beta\alpha a + \beta a .$$

This conjunction is equivalent to the relationship $\beta = \alpha^{-1}$. Thus, condition *1)* holds for an arbitrary quasigroup $(Q;\cdot)$ satisfying (i). Let $\mathscr{S}$ denote the variety defined by (i). The quasigroup $(Q;\cdot)$ belongs to $\mathscr{S}$ and $x \cdot y = \alpha x + a + \alpha^{-1} y$ is its canonical decomposition. By Theorem 4, $(Q;\cdot)$ is medial. Since an arbitrary quasigroup of $\mathscr{S}$ is medial, then the variety $\mathscr{S}$ is a subvariety of the variety of medial quasigroups.

2) Let $(Q;\cdot)$ be a quasigroup satisfying (vii). Replacing x with $v \overset{r}{\cdot} x$ and u with $u \overset{l}{\cdot} v$ and using the second and the fourth identities from (1), we obtain

$$(v \overset{r}{\cdot} x)\cdot yu = xy\cdot(u \overset{l}{\cdot} v).$$

Since $(Q;\cdot)$ is group isotope, then by Corollary 5, selecting the variables x, u we conclude that canonical decomposition group is commutative. Consider the identity (x). Replace the operation $(\cdot)$ with $(+)$ according to (2):

$$\alpha x + a + \beta(\alpha y + a + \beta(\alpha u + a + \beta v)) = \alpha(\alpha(\alpha v + a + \beta x) + a + \beta y) + a + \beta v. \qquad (6)$$

Put $u = v = 0$ in (6) then by Corollary 7, α is an automorphism. Similarly, from $x = v = 0$ transformation β is an automorphism. Then (6) has the form

$$\alpha x + a + \beta\alpha y + \beta\alpha + \beta^2\alpha u + \beta^2\alpha + \beta^3 v = \alpha^3 y + \alpha^2 a + \alpha^2\beta x + \alpha a + \alpha\beta y + a + \beta u.$$

By virtue of Lemma 8, the conjunction

$$\alpha = \alpha^2\beta, \quad \beta\alpha = \alpha\beta, \quad \beta^2\alpha = \beta, \quad \beta^3 = \alpha^3, \quad a + \beta a + \beta^2 a = \alpha^2 a + \alpha a + a$$

is equivalent to the last equality. This conjunction can be written as follows:

$$\beta = \alpha^{-1}, \qquad \alpha^6 = \iota, \qquad \alpha^2 a + \alpha a - \alpha^{-1}a - \alpha^{-2}a = 0.$$

The third equality is equivalent to $\alpha^4 a + \alpha^3 a - \alpha a - a = 0.$ Thus, we obtain conditions *2)*. Variety $\mathfrak{V}_1$ which is defined by the identity (vii) is subvariety of the variety of medial quasigroups, since conditions *2)* satisfy Theorem 4.

3) Let $(Q;\cdot)$ be a quasigroup satisfying (x). Replacing y with $x \overset{r}{\cdot} y$ and v with $v \overset{l}{\cdot} u$ and using the second and the fourth identities from (1), we obtain

$$yu\cdot xv = (x \overset{r}{\cdot} y)\cdot(v \overset{l}{\cdot} u).$$

Since $(Q;\cdot)$ is a group isotope, then by Corollary 5 selecting the variables x, u, we conclude commutativity of its canonical decomposition group. Consider the identity (x). In (x) according to (2), we replace the operation $(\cdot)$ with $(+)$:

$$\alpha(\alpha(\alpha x + a + \beta y) + a + \beta u) + a + \beta(\alpha x + a + \beta(\alpha v + a + \beta u)) = \alpha y + a + \beta v. \qquad (7)$$

Put $y = u = 0$ in (7), then according to Corollary 7, β is an automorphism, so the identity (7) can be written as follows:

$$\alpha(\alpha(\alpha x + a + \beta y) + a + \beta u) + a + \beta\alpha x + \beta a + \beta^2\alpha v + \beta^2 a + \beta^3 u = \alpha y + a + \beta v.$$

Put $x = v = 0$ in the last identity then by Corollary 7, α is an automorphism and the identity (7) has the form

$$\alpha^3 x + \alpha^2 a + \alpha^2\beta y + \alpha a + \alpha\beta u + a + \beta\alpha x + \beta a + \beta^2\alpha v + \beta^2 a + \beta^3 u = \alpha y + a + \beta v.$$

According to Lemma 8, it is equivalent to

$$\alpha^3 + \beta\alpha = 0, \quad \alpha^2\beta = \alpha, \quad \alpha\beta + \beta^3 = 0, \quad \beta^2\alpha = \beta, \quad \alpha^2 a + \alpha a + a + \beta a + \beta^2 a = a.$$

This conjunction is equivalent to the conditions *3)*, since the second equality of conjunction implies $\beta = \alpha^{-1}$ and the third equality $\alpha = -\beta^2 = -\alpha^2$, i.e., $\alpha^3 = -\iota$. If the identity (x) defines a variety $\mathfrak{V}_2$, then $\mathfrak{V}_2$ is subvariety of the variety of medial quasigroups, because conditions *3)* satisfy Theorem 4.

All transformations of the proof are equivalent. That is why the theorem has been proved. □

Note. Coefisients of the canonical conditions of the identity (i) and coefisients of semisymmetric group isotopes coincide. The canonical conditions of (i) do not have any conditions for their coefisients and free members.

The following two lemmas describe connections among canonical conditions of (i), (vii) and canonical conditions of their σ-parastrophes.

Lemma 14. *Let* α *be a left coefficient of the canonical conditions of* (i), *then canonical conditions of the identities* (iii) *and* (iv) *have the following forms:*

1) $x \circ y = \alpha^{-1}x - \alpha^{-1}a - \alpha^{-2}y$ *for the identity* (iii);

2) $x \circ y = -\alpha^2 x - \alpha a + \alpha y$ *for the identity* (iv).

Proof. 1) Let $\mathfrak{V}$ be a variety of a quasigroup defined by the identity (i). According to the definition, a quasigroup $(Q;\cdot)$ belongs to the variety ${}^{l}\mathfrak{V}$ if and only if its l-parastrophe $(Q;\overset{l}{\cdot})$ belongs to the variety $\mathfrak{V}$. By

p. *1)* of Theorem 5, this is equivalent to the existence of the commutative group $(Q;+)$ and to its automorphism α such that p. *1)* of Theorem 5 holds. Then by virtue of (4), we obtain

$$x \overset{l}{\cdot} y = \alpha^{-1}x - \alpha^{-1}a - \alpha^{-2}y\,. \tag{8}$$

By Lemma 11, (iii) is l-parastrophe of the identity (i) then (8) is the canonical decomposition of quasigroups which satisfy (iii). Denoting $(\overset{l}{\cdot}) =: (\circ)$, (8) expresses the canonical conditions of the identity (iii), i.e., p. *1)* of this lemma holds.

2) In the same way, one can prove that

$$x \overset{r}{\cdot} y = -\alpha^{2}x - \alpha a + \alpha y$$

is canonical condition of (iv) for $(\overset{r}{\cdot}) =: (\circ)$, that is the second condition of this lemma is true. □

Lemma 15. *Let α be a left coefficient of the canonical conditions of (vii), then canonical conditions of the identities (viii) and (ix) satisfy canonical conditions of (vii) and have the following forms:*

1) $x \circ y = \alpha^{-1}x - \alpha^{-1}a - \alpha^{-2}y$ *for the identity* (viii);

2) $x \circ y = -\alpha^{2}x - \alpha a + \alpha y$ *for the identity* (ix).

Proof. The lemma is proved similarly to Lemma 14. In particular, conditions *2)* of Theorem 5 satisfy canonical conditions of (viii) and (ix), since they hold for (vii) and α is the first component of the canonical conditions of the identity (vii). □

4. Varieties of quasigroups and their symmetry

According to the symmetry concept, trusses of varieties defined by given identities are characterized. Quasigroups from each of these varieties are described according to their symmetry groups and the belonging to well-known classes.

4.1. Symmetry of parastrophic varieties

Let $\mathcal{A}$ be a variety of quasigroups, then ${}^{\sigma}\mathcal{A}$ is a σ-parastrophe of $\mathcal{A}$. If $\mathrm{Sym}\,\mathcal{A} = S_3$, then variety is called *totally symmetric*; if $\mathrm{Sym}\,\mathcal{A} = \{\iota, s\}$, i.e., $\mathcal{A}={}^{s}\mathcal{A}$, then $\mathcal{A}$ is *middle symmetric*; if $\mathrm{Sym}\,\mathcal{A} = \{\iota, l\}$, i.e., $\mathcal{A}={}^{l}\mathcal{A}$, then $\mathcal{A}$ is *right symmetric*; if $\mathrm{Sym}\,\mathcal{A} = \{\iota, r\}$, i.e., $\mathcal{A}={}^{r}\mathcal{A}$, then $\mathcal{A}$ is *left symmetric*.

A set of all pairwise parastrophic classes is called a *truss*. The notion of a truss is introduced by F. Sokhatsky [13]. A truss of varieties is uniquely defined by one of its varieties. The number of different varieties being parastrophic to $\mathcal{A}$ is $6/|\mathrm{Sym}\,\mathcal{A}|$, that is 1, 2, 3 or 6. Consequently, a truss is considered a one-element, a two-element, a three-element or a six-element one.

The following theorems describe trusses of varieties.

Theorem 6. *Each of identities* (i)-(vi) *defines the same three-element truss of varieties. Moreover, if a variety $\mathcal{B}$ is defined by the identity* (i)*, then* (ii) *also defines $\mathcal{B}$, identities* (iii) *and* (vi) *define a variety ${}^{l}\mathcal{B}$, identities* (iv) *and* (v) *define a variety ${}^{r}\mathcal{B}$.*

Proof. Let $\mathcal{B}$ denote the variety being defined by (i). Let $(Q;\cdot)$ be a quasigroup from $\mathcal{B}$ and

$$x \cdot y = \alpha x + a + \alpha^{-1}y$$

be its canonical decomposition (Theorem 5). The equalities (4) imply

$$x \overset{s}{\cdot} y = \alpha^{-1}x + a + \alpha y$$

which is a canonical decomposition of s-parastrophe of $(Q;\cdot)$. Since $(\alpha^{-1})^{-1} = \alpha$, then $(Q;\overset{s}{\cdot})$ belongs to $\mathcal{B}$. Therefore ${}^{s}\mathcal{B} = \mathcal{B}$. This equality implies

$${}^{l}({}^{s}\mathcal{B})={}^{l}\mathcal{B} \qquad \text{and} \qquad {}^{r}({}^{s}\mathcal{B})={}^{r}\mathcal{B},$$

i.e.,

$${}^{sr}\mathcal{B}={}^{l}\mathcal{B} \qquad \text{and} \qquad {}^{sl}\mathcal{B}={}^{r}\mathcal{B}.$$

Taking into account Lemma 11 and the obtained equalities, a variety ${}^{l}\mathcal{B}$ is defined by one of the identities (iii) and (vi), ${}^{r}\mathfrak{B}$ by one of (iv) and (v).

It is well known that $\{s,l\}$ generates symmetry group S_3. It means that the coincidence of s - and l - parastrophes of a variety implies its totall symmetry. To prove that the truss $\{\mathfrak{B}, {}^{l}\mathfrak{B}, {}^{r}\mathfrak{B}\}$ is three-element it is enough to show that $\mathfrak{B}$ and ${}^{l}\mathfrak{B}$ are different. For this purpose, we consider a quasigroup $(Z_{11};\circ)$, where Z_{11} is a ring modulo 11 and

$$x \circ y = 3x + 1 + 4y.$$

Since $3^{-1} = 4$, then by Theorem 5, $(Z_{11};\circ)$ belongs to $\mathfrak{B}$. The equalities (4) imply

$$x \overset{l}{\circ} y = 3^{-1}(x - 1 - 4y) = 4x - 4 - 16y = 4x + 7 + 6y.$$

Since $4^{-1} = 3 \neq 6$, then $(Z_{11};\overset{l}{\circ})$ does not belong to $\mathfrak{B}$, but belongs to ${}^{l}\mathfrak{B}$. That is why $\mathfrak{B} \neq {}^{l}\mathfrak{B}$.

Thus, the variety $\mathfrak{B}$ is middle symmetric and the corresponding truss is three-element. □

Corollary 16. *The varieties* $\mathfrak{B}$, ${}^{l}\mathfrak{B}$ *and* ${}^{r}\mathfrak{B}$ *are middle symmetric, left symmetric and right symmetric respectively.*

Proof. The proof follows from Theorem 6 and Corollary 3, since $\mathrm{Sym}\mathfrak{B} = \{\iota, s\}$ and

$$\mathrm{Sym}({}^{l}\mathfrak{B}) = l(\mathrm{Sym}\mathfrak{B})l = l\{\iota, s\}l = \{l\iota l, lsl\} = \{\iota, r\},$$

$$\mathrm{Sym}({}^{r}\mathfrak{B}) = r(\mathrm{Sym}\mathfrak{B})r = r\{\iota, s\}r = \{r\iota r, rsr\} = \{\iota, l\}.$$ □

Theorem 7. *Each of identities* (vii)-(ix) *defines the same three-element truss of varieties. Moreover, if a variety* $\mathfrak{B}_1$ *is defined by the identity* (vii), *then identities* (viii) *and* (ix) *define varieties* ${}^{l}\mathfrak{B}_1$ *and* ${}^{r}\mathfrak{B}_1$ *respectively.*

Proof. Let $\mathfrak{B}_1$ denote the variety being defined by (vii). Lemma 12 implies ${}^{s}\mathfrak{B}_1 = \mathfrak{B}_1$ and the varieties ${}^{l}\mathfrak{B}_1$, ${}^{r}\mathfrak{B}_1$ are defined by (viii), (ix) respectively. Similarly to Theorem 6, the truss of varieties $\mathfrak{B}_1$, ${}^{l}\mathfrak{B}_1$, ${}^{r}\mathfrak{B}_1$ is three-element, if $\mathfrak{B}_1$ and ${}^{l}\mathfrak{B}_1$ are different. Consider a quasigroup $(Z_9;*)$ being defined by

$$x * y = 4x + 1 + 7y.$$

Since $4^{-1} = 7$, $4^6 = 1$ and $4^4 \cdot 1 + 4^3 \cdot 1 - 4 \cdot 1 - 1 = 0$, then by Theorem 5, $(Z_9;*)$ belongs to $\mathfrak{B}_1$. (4) implies

$$x \overset{l}{*} y = 4^{-1}(x - 1 - 7y) = 7x - 7 - 49y = 7x + 2 + 5y.$$

Because $7^{-1} = 4 \neq 5$, then $(Z_9;\overset{l}{*})$ does not belong to $\mathfrak{B}_1$, but belongs to ${}^{l}\mathfrak{B}_1$. It means $\mathfrak{B}_1 = {}^{l}\mathfrak{B}_1$.

So, the variety $\mathfrak{B}_1$ is middle symmetric and the truss, which is defined by $\mathfrak{B}_1$ is three-element. □

Corollary 17. *The varieties* $\mathfrak{B}_1$, ${}^{l}\mathfrak{B}_1$ *and* ${}^{r}\mathfrak{B}_1$ *are middle symmetric, left symmetric and right symmetric respectively.*

Proof. The proof is the same as that of Corollary 16. □

Theorem 8. *The identity* (x) *defines a totally symmetric variety, i.e.,* (x) *defines a one-element truss.*

Proof. The proof follows from Lemma 13. □

4.2. Description of quasigroups of varieties

In this subsection, taking into account given results and Corollary 10, we obtain partition of group isotopes of varieties $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ and their parastrophes.

Proposition 18. *Quasigroups satisfying the identities* (i), (vii), (x) *are groups if and only if coefficients of their canonical decompositions are identical permutations.*

Proof. By Proposition 9, the statement is true. □

Remark 1. *The condition of Proposition 19 implies that groups, which satisfy the identity* (x) *are Boolean. Indeed, when* $\alpha = \iota$ *taking into account condition* $\alpha^3 = -\iota$ *of its canonical conditions, then* $\iota = -\iota$. *It means that these groups are Boolean.*

The following two corollaries describe all quasigroups, which belong to the varieties $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ and establish necessary and sufficient conditions of belonging a quasigroup exactly to one of the selected classes, which are gived in brackets. Similarly to H.V. Krainichuk [12], we use the word ‘strictly’ to emphasize that the class does not cointain any totally symmetric quasigroup.

Corollary 19. *All quasigroups from varieties $\mathcal{S}$ and $\mathcal{S}_1$ are distributed into five disjoint classes:*

1) *groups ($\alpha = \iota$);*
2) *totally symmetric quasigroups ($\alpha = -\iota$);*
3) *strictly commutative quasigroups ($\alpha^2 = \iota$ and $-\iota \neq \alpha \neq \iota$);*
4) *strictly semisymmetric quasigroups ($\alpha^3 = -\iota$, $\alpha a = -a$ and $\alpha \neq -\iota$);*
5) *asymmetric quasigroups ($\alpha^2 \neq -\iota$ and ($\alpha^3 \neq -\iota$ or $\alpha a \neq -a$)).*

Proof. The proof is immediately follows from Corollary 10 and Proposition 18. □

Note. The classes of strictly commutative and strictly semisymmetric quasigroups of the varieties $\mathcal{S}$ and $\mathcal{S}_1$ are empty under the additive group of a field. But there exist examples of such quasigroups in rings.

Example 2. Consider the quasigroups $(Z_8;\otimes)$ and $(Z_9;\oplus)$ defined by

$$x \otimes y = 3x + 3y \qquad \text{and} \qquad x \oplus y = 2x + 3 + 5y\,.$$

They both belong to $\mathcal{S}$ and $\mathcal{S}_1$. $(Z_8;\otimes)$ is a strictly commutative quasigroup, since $3^2 = 9 = 1$ and $-1 \neq 3 \neq 1$, $(Z_9;\oplus)$ is a strictly semisymmetric because $2^3 = 8 = -1$, $2 \cdot 3 = 6 = -3$ and $2 \neq -1$.

The last corollary confirms that the classes of the quasigroups of varieties $\mathcal{S}$ and $\mathcal{S}_1$ which are described in pp. 1)-4) coincide. The variety $\mathcal{S}_1$ is included in the variety $\mathcal{S}$ and they differ from each other by asymmetric quasigroups, in particular, the following example shows that the varieties $\mathcal{S}$ and $\mathcal{S}_1$ are diferent.

Example 3. The quasigroup $(Z_{11};\bullet)$, where $x \bullet y = 3x + 1 + 4y$ belongs to $\mathcal{S}$, but does not belong to $\mathcal{S}_1$, since $3^6 = 8 \neq 1$.

Corollary 20. *All quasigroups from variety $\mathcal{S}_2$ are distributed into five disjoint classes:*

1) *boolean groups ($\alpha = \iota$);*
2) *nonboolean totally symmetric quasigroups ($-\iota \neq \alpha \neq \iota$);*
3) *strictly semisymmetric quasigroups ($\alpha a \neq -a$ and $\alpha \neq -\iota$);*
4) *asymmetric quasigroups ($\alpha^2 \neq -\iota$ and $\alpha a \neq -a$).*

Proof. By virtue of Corollary 10 and Remark 1, the corollary is evident. □

The next two corollaries describe quasigroups of the parastrophic varieties ${}^l\mathcal{S}$, ${}^r\mathcal{S}$, ${}^l\mathcal{S}_1$, ${}^r\mathcal{S}_1$.

Corollary 21. *All quasigroups from varieties ${}^l\mathcal{S}$ and ${}^l\mathcal{S}_1$ are distributed into three disjoint classes:*

1) *totally symmetric quasigroups ($\alpha = \iota$);*
2) *strictly left symmetric quasigroups ($\alpha^2 = \iota$ and $\alpha \neq \iota$);*
3) *asymmetric quasigroups ($\alpha^2 \neq \iota$).*

Proof. The proof is immediately follows from Corollary 10. □

Corollary 22. *All quasigroups from varieties ${}^r\mathcal{S}$ and ${}^r\mathcal{S}_1$ are distributed into three disjoint classes:*

4) *totally symmetric quasigroups ($\alpha = -\iota$);*
5) *strictly right symmetric quasigroups ($\alpha^2 = \iota$ and $\alpha \neq -\iota$);*
6) *asymmetric quasigroups ($\alpha^2 \neq \iota$).*

Proof. The proof follows from Corollary 10. □

Corollary 23. *The relationships*

$$\mathcal{S} \supset \mathcal{S}_1, \qquad {}^l\mathcal{S} \supset {}^l\mathcal{S}_1, \qquad {}^r\mathcal{S} \supset {}^r\mathcal{S}_1,$$
$$\mathcal{S} \supset \mathcal{S}_2, \qquad {}^l\mathcal{S} \supset \mathcal{S}_2, \qquad {}^r\mathcal{S} \supset \mathcal{S}_2$$

hold for varieties $\mathcal{S}$, $\mathcal{S}_1$, $\mathcal{S}_2$ and their parastrophes.

Proof. The inclusion $\mathcal{S} \supset \mathcal{S}_1$ follows from Corollary 19 and Example 2. Since in Theorem 5 conditions *3)* satisfy conditions *1)*, then $\mathcal{S} \supset \mathcal{S}_2$. The quasigroup $(Z_{11};\bullet)$ from Example 2 illustrates the correctness of strict inclusion, since it belongs to the variety $\mathcal{S}$ and does not belong to $\mathcal{S}_2$. The relationships for parastrophes of $\mathcal{S}$ and $\mathcal{S}_1$ follow from Proposition 4. Since $\mathcal{S}_2 = {}^\sigma\mathcal{S}_2$ for all $\sigma \in S_3$, then ${}^l\mathcal{S} \supset \mathcal{S}_2$, ${}^r\mathcal{S} \supset \mathcal{S}_2$. □

Corollary 23. *The semisymmetry-like variety* $\mathcal{S}$ *and its subvarieties* $\mathcal{S}_1$, $\mathcal{S}_2$ *contain the class of all semisymmetry isotopes of commutative groups.*

Proof. By Corollary 10 and Theorem 5, easy to verify that the statement of the corollary holds. □

Conclusions.

The results of this article are systematized in the following table, whose first column contains the trusses of varieties, the second one includes the varieties belonging to the corresponding truss, the third one provides the identities, which define these varieties (equivalent identities are in the same row), the fourth one includes the canonical conditions of the identities. In the last row the relationships among all varieties are given.

Truss	Variety	Identity	Canonical conditions
$\mathcal{S}$	$\mathcal{S}={}^{s}\mathcal{S}$	(i), (ii)	$x\cdot y=\alpha x+a+\alpha^{-1}y$
	${}^{l}\mathcal{S}={}^{sr}\mathcal{S}$	(iii), (vi)	$x\overset{l}{\cdot}y=\alpha^{-1}x-\alpha^{-1}a-\alpha^{-2}y$
	${}^{r}\mathcal{S}={}^{sl}\mathcal{S}$	(iv), (v)	$x\overset{r}{\cdot}y=-\alpha^{2}x-\alpha a+\alpha y$
$\mathcal{S}_1$	$\mathcal{S}_1={}^{s}\mathcal{S}_1$	(vii)	$x\cdot y=\alpha x+a+\alpha^{-1}y,\ \alpha^{6}=\iota,\ \alpha^{4}a+\alpha^{3}a-\alpha a-a=0$
	${}^{l}\mathcal{S}_1={}^{sr}\mathcal{S}_1$	(viii)	$x\overset{l}{\cdot}y=\alpha^{-1}x-\alpha^{-1}a-\alpha^{-2}y,\ \alpha^{6}=\iota,\ \alpha^{4}a+\alpha^{3}a-\alpha a-a=0$
	${}^{r}\mathcal{S}_1={}^{sl}\mathcal{S}_1$	(x)	$x\overset{r}{\cdot}y=-\alpha^{2}x-\alpha a+\alpha y,\ \alpha^{6}=\iota,\ \alpha^{4}a+\alpha^{3}a-\alpha a-a=0$
$\mathcal{S}_2$	$\mathcal{S}_2={}^{\sigma}\mathcal{S}_2$, $\forall\sigma\in S_3$	(x)	$x\cdot y=\alpha x+a+\alpha^{-1}y,\ \alpha^{3}=-\iota$
	$\mathcal{S}\supset\mathcal{S}_1$,	${}^{l}\mathcal{S}\supset{}^{l}\mathcal{S}_1$,	${}^{r}\mathcal{S}\supset{}^{r}\mathcal{S}_1$, $\mathcal{S}\supset\mathcal{S}_2$, ${}^{l}\mathcal{S}\supset\mathcal{S}_2$, ${}^{r}\mathcal{S}\supset\mathcal{S}_2$

The canonical conditions of the identities are constructive and allow to build the quasigroups of all described varieties quite easily and to study their spectrum of quasigroups. It is necessary to find all identities defining semisymmetry-like medial quasigroups and to describe relationships among the corresponding varieties.

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МНОГОВИД НАПІВ СИМЕТРИЧНО-ПОДІБНИХ МЕДІАЛЬНИХ КВАЗІГРУП ТА ЙОГО ПІДМНОГОВИДИ

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РЕЗЮМЕ

У цій статті описано тотожності, які визначають многовиди $\mathcal{S}$, $\mathcal{S}_1$, $\mathcal{S}_2$, подібні до многовида напівсиметричних ізотопів групи. Встановлено мови, за яких квазігрупи з $\mathcal{S}$, $\mathcal{S}_1$ і $\mathcal{S}_2$ збігаються із напівсиметричними ізотопами комутативної групи. Згідно принципу симетрії, описано ці три многовиди та їх парастрофні многовиди і квазігрупи, які належать усім цим многовидам. Показано співвідношення між усіма цими многовидами.

Ключові слова: квазігрупа, тотожність, парастроф, симетрія, ізотоп групи, многовид, медіальний, асиметричний, комутативний, напів симетричний, тотально-симетричний.

МНОГООБРАЗИЕ ПОЛУ СИММЕТРИЧНО-ПОДОБНЫХ МЕДИАЛЬНЫХ КВАЗИГРУПП И ЕГО ПОДМНОГООБРАЗИЯ

О.О. Тарковская

РЕЗЮМЕ

В этой статье описаны тождества, которые определяют многообразия $\mathcal{S}$, $\mathcal{S}_1$, $\mathcal{S}_2$, подобные многообразию полусимметрических групповых изотопов. Установлены условия, при которых квазигруппы из $\mathcal{S}$, $\mathcal{S}_1$, $\mathcal{S}_2$ совпадают с полусимметрическими ізотопами коммутативной группы. Согласно принципу симметрии, описаны эти три многообразия и многообразия, парастрофны им, и квазигруппы принадлежащие всем этим многообразиям. Показаны соотношения между всеми этими многообразиями.

Ключевые слова: квазигруппа, тождество, парастроф, симметрия, изотоп группы, многообразие, медиальный, ассиметричный, коммутативный, полу симметричный, тотально-симметриный.