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NONLOCAL BOUNDARY-VALUE FOR ABSTRACT SECOND-ORDER DIFFERENTIAL EQUATION WITH OPERATOR INVOLUTION

We study a nonlocal problem with generalized conditions Ionkin's for the Sturm-Liouville equation with polynomial potential which contains an involution operator. The spectral properties of the operator of this problem are analyzed and the conditions for the existence and uniqueness of its solution are established. It is also proved that the system of root functions essentially a nonself-adjoint operator of the analyzed problem forms a Riesz basis.

Keywords: *differential-operator equation, root function, operator of involution, essentially a nonself-adjoint operator, Riesz basis, nonlocal problem*

Introduction

Purpose that H is a separable Hilbert space. $L(H)$ is the algebra of bounded linear operators $H \to H$. $A : H \to H$ is a positive self-adjoint operator with point spectrum

$$\sigma_p(A) = \{z_k \in \mathbb{R} : z_k \sim \beta k^{\alpha},\ \alpha, \beta > 0,\ k = 1, 2, \ldots, \}.$$

$V(A) \equiv \{v_k \in H, k = 1, 2, \ldots\}$ is a system of eigenfunctions that form an orthonormal basis in the space H. $H(A^s) = \{v \in H : A^s v \in H\}$, $s \geqslant 0$, $(u, v; H(A^s)) \equiv (A^s u, A^s v; H)$, $\|y; H(A^s)\|^2 = \|A^s y; H\|^2$, $s \geqslant 0$; $H_1 \equiv \equiv L_2((0,1), H)$; $\|u; H_1\|^2 = \left\| \|u; H\|^2; L_2(0,1) \right\|$. $D_x : H_1 \to H_1$ is a strong derivative in the space H_1; $\left\| \frac{u(x+\triangle x) - u(x)}{\triangle x} - D_x u; H_1 \right\| \to 0, \triangle x \to 0$. $I : L_2(0,1) \to L_2(0,1)$ is an involution operator: $Iu(x) \equiv u(1-x)$, $u(x) \in L_2(0,1)$; $p_0 = \frac{1}{2}(E + I), p_1 = \frac{1}{2}(E - I)$ is the ortoproektor in $L_2(0,1)$, $L_{2,j}(0,1) \equiv \{y(x) \in L_2(0,1) : y(x) = p_j y(x)\}$, $j = 0, 1$; $p_{1,j} = p_j \bigotimes E_H$, $E_H u \equiv u, u \in H, j = 0, 1$; $H_{1,j} \equiv \{y(x) \in H_1 : y(x) \equiv p_{1,j} y(x)\}$, $j = 0, 1$, $H_2 \equiv \left\{y(x) \in H_1; D_x^2 y \in H_1, A^2 y \in H_1\right\}$, $\|y; H_2\|^2 \equiv \left\|D_x^2 y; H_1\right\|^2 + \left\|A^2 y; H_1\right\|^2$. $L(H(A^m); H(A^q))$ is the algebra of bounded linear operators $A : H(A^m) \to H(A^q)$, $m, q \geqslant 0$; $L(H(A^m)) \equiv L(H(A^m); H(A^m))$, $B_1, B_2 \in L(H)$, $B_0 \in L(H_1)$, $B_r v_k = b_{r,k} v_k$, $b_{r,k} \in \mathbb{R}$, $r = 0, 1, 2$, $k = 1, 2, \ldots$; $H^1 \equiv H(A^{\frac{3}{2}})$, $H^2 \equiv H(A^{\frac{1}{2}})$.

Consider the following problem:

$$L(D_x, A)y \equiv -D_x^2 y(x) + A^2 y(x) + B_0(2x-1)(y(x) + y(1-x)) = f(x), \tag{1}$$

$$\ell_1 y \equiv B_1 y(0) + B_2 y(1) = h_1,\ \ell_2 y \equiv D_x y(0) - D_x y(1) = h_2, \tag{2}$$

$f(x) \in H_1$, $h_1 \in H^1$, $h_2 \in H^2$.

We interpret the solution [13, 18] of problem (1), (2) as a function $y(x) \in H_2$, satisfying the equalities

$$\|L(D_x, A)y - f; H_1\| = 0,\ \left\|\ell_1 y - h_1; H^1\right\| = \left\|\ell_2 y - h_2; H^2\right\| = 0. \tag{3}$$

Differential equation (1) includes operator of involution $I : L_2(0,1) \to L_2(0,1)$, $Iy(x) \equiv y(1-x)$, $x \in L_2(0,1)$.

The first time study properties of the operator involution started C. Babbage [3]. In the paper [10] T. Carleman introduced the concept of operator shift – a generalization of the concept of involution $Iu(x) \equiv u(1-x), (u(x) \in L_2(0,1))$. Exploration partial differential equations with involution are devoted [2, 5, 9, 10, 19, 20, 34].

Properties spectral problems for ordinary differential and functional-differential equations with involution investigated in the works [7, 8], [14]–[18], [21],[22], [26]–[29], [35] and [3, 4, 6, 25] respectively.

V. A. Il'in in [15]–[17] introduced the concept of essentially nonself-adjoint operator and proved the existence of present criterion of root functions of spectral problems for differential equations of arbitrary order.

In 1979 A.A. Shkalikov [30] it was proved that Riesz basis of subspaces (block basis) to $L_2(0,1)$ root system forms a subspace that meet multiple or asymptotically close eigenvalues of differential operator A.

In works [31]–[33] for equation Sturm-Liouville studied the spectral properties of problem with linear potential with Dirichlet conditions in the interval $(0,1)$.

In works [1], [23], [24] studied the properties of operators with increasing the multiplicity of the spectrum.

Main part

Auxiliary spectral problems

We now consider $L : H_1 \to H_1$ is the operator of problem (1), (2):

$$Ly \equiv L(D_x, A)y, y \in D(L),\ D(L) \equiv \{y \in H_2; \ell_1 y = 0,\ \ell_2 y = 0\}.$$

Solutions of spectral problem

$$L(D_x, A)y \equiv -D_x^2 y(x) + A^2 y + 2B_0(D_x y(x) + D_x y(1-x)) = \lambda y(x), \lambda \in \mathbb{C}, \tag{4}$$

$$\ell_1 y \equiv B_1 y(0) + B_2 y(1) = 0,\ \ell_2 y \equiv D_x y(0) - D_x y(1) = 0 \tag{5}$$

consider as a product $y(x) = u(x)v_k,\ u(x) \in W_2^2(0,1),\ k = 1, 2, \ldots$.

To determine the functions $u(x)$ obtain spectral problem

$$L_k(D_x)u \equiv -D_x^2 u(x) + z_k^2 u(x) + 2b_{0,k}(2x-1)(u(x) + u(1-x)) = \lambda u(x), \tag{6}$$

$$\ell_{1,k} u \equiv b_{1,k} u(0) + b_{2,k} u(1) = 0,\ \ell_{2,k} u \equiv D_x u(0) - D_x u(1) = 0. \tag{7}$$

Consider the particular case the problem $B_1 = -B_2 = E,\ B_0 = 0,$ if the specified conditions (6), (7)

$$-D_x^2 u(x) + z_k^2 u = \lambda u(x), \tag{8}$$

$$u(0) - u(1) = 0,\ D_x u(0) - D_x u(1) = 0. \tag{9}$$

Theorem 1. *Let* $B_2 = -B_1 = E$, $B_0 = 0$. *Then problem* (8),(9) *have point spectrum*

$$\sigma_k = \left\{\lambda_{k,n} \in \mathbb{R} : \lambda_{k,n} = (2\pi n)^2 + z_k^2, n = 0, 1, \ldots\right\},$$

and system of eigenfunctions

$$T \equiv \left\{t_n^s \in L_2(0,1) : t_0^0(x) = 1,\ t_n^0(x) = \sqrt{2}\cos 2\pi n x,\ t_n^1(x) = \sqrt{2}\sin 2\pi n x,\ n \in \mathbb{N}\right\}.$$

We now consider $L_{0,k} : L_2(0,1) \to L_2(0,1)$ is the operator of the problem (8), (7)

$$L_{0,k} u \equiv (-D_x^2 + z_k^2)u,\ u \in D(L_{0,k}),$$

$$D(L_{0,k}) \equiv \left\{u \in W_2^2(0,1) : \ell_{1,k} u = 0, \ell_{2,k} u = 0\right\}.$$

Let

$$v_{k,0}^{0,0}(x) = 1 + \beta_{k,n}(2x-1),\ v_{k,n}^{0,0}(x) = \sqrt{2}\sin 2\pi n x, \tag{10}$$

$$v_{k,n}^{1,0}(x) = \sqrt{2}(1 + \beta_{k,n}(2x-1))\cos 2\pi n x, \tag{11}$$

$$\beta_{k,n} \equiv (b_{1,k} - b_{2,k})^{-1}(b_{1,k} + b_{2,k}),\ k, n = 1, 2, \ldots . \tag{12}$$

You can check that

$$L_{0,k} v_{k,n}^{1,0}(x) = \lambda_{k,n} v_{k,n}^{1,0}(x) + \xi_{k,n}^0 v_{k,n}^{0,0}(x),\ \xi_{k,n}^0 = 8\pi n \beta_{k,n}, \tag{13}$$

$k = 1, 2, \ldots$.

Hence, $V(L_{0,k}) \equiv \left\{v_{k,n}^{s,0}(x), s = 0, 1,\ n = 0, 1, \ldots\right\}$ is the system of root functions of the operator in the sense of equality (13).

Theorem 2. *Let $b_{1,k} \neq b_{2,k}$. Then the operator $L_{0,k}$ of problem (9), (8) have point spectrum σ_k and system $V(L_{0,k})$ of root functions complete and minimal in $L_2(0,1)$.*

Proof. We now prove the completeness of system $V(L_{0,k})$ in the space $L_2(0,1)$.
Consider the adjoint problem

$$-D_x^2 v(x) + z_k^2 v = \overline{\lambda} u(x),\ v(0) - v(1) = 0,\ b_{2,k} D_x v(0) + b_{1,k} D_x v(1) = 0.$$

The operator of this problem have point spectrum σ_k and system of root functions

$$W(L_{0,k}) \equiv \Big\{ w_{k,n}^{s,0} \in L_2(0,1) : w_{k,q}^{0,0} = \sqrt{2}\cos 2\pi q x,$$

$$w_{k,q}^{1,0} = \sqrt{2}(1 - \beta_{k,q}(2x-1))\sin 2\pi q x,\ k = 1,2,\ldots \Big\}.$$

Hence, the system of root functions $V(L_{0,k})$ of the operator $L_{0,k}$ possesses a unique biorthogonal system $W(L_{0,k})$

$$(v_{k,n}^{r,0}, w_{k,q}^{s,0}; L_2(0,1)) = \delta_{r,s}\delta_{n,q},\ r,s = 0,1,\ q,n = 1,2,\ldots.$$

□

Consider the operators $R_{0,k}, S_{0,k} : L_2(0,1) \to L_2(0,1)$,

$$R_{0,k} t_{k,n}^{p} \equiv v_{k,n}^{p,0},\ R_{0,k} = E + S_{0,k},\ p = 0,1,\ n = 0,1,\ldots.$$

From the definition of the operator $R_{0,k}$ and the completeness of system $V(L_{0,k})$ in space $L_2(0,1)$, we get $S_{0,k} : L_{2,0}(0,1) \to L_{2,1}(0,1)$, $S_{0,k} : L_{2,1}(0,1) \to 0$. Then, $S_{0,k}S_{0,k} = O$, where O is the zero operator in the $L_2(0,1)$.

Thus, $R_{0,k}^{-1} = E - S_{0,k}$.

To prove that the system $V(L_{0,k})$ forms a Riesz basis [12] in $L_2(0,1)$, it is sufficient, according to formula $R_{0,k} = E + S_{0,k}$, to show that the operator $S_{0,k} : L_2(0,1) \to L_2(0,1)$ is bounded.

Let ω be an arbitrary element from the space $L_2(0,1)$. We represent ω as a Fourier series in the system T

$$\omega = \omega_0^0 t_0^0 + \sum_{m=1}^{\infty} \omega_m^0 t_m^0 + \omega_m^1 t_m^1,\ \omega_m^j = \left(\omega, t_m^j; L_2(0,1)\right),\ j = 0,1,\ m = 0,1,\ldots.$$

According to the definition of the operator $S_{0,k}$, we find

$$S_{0,k}\omega = \beta_{k,0}(2x-1)\left(\omega_0^0 v_{k,0}^0 + \sum_{m=1}^{\infty} \omega_m^0 \sqrt{2}\cos 2\pi m x\right).$$

Using the ratio

$$\|S_{0,k}\omega; L_2(0,1)\|^2 =$$

$$= \left\|\beta_{k,0}(2x-1)\left(\omega_0^0 v_{k,0}^0 + \sum_{m=1}^{\infty} \omega_m^0 \sqrt{2}\cos 2\pi m x\right); L_2(0,1)\right\|^2,$$

we estimate it

$$\|S_{0,k}\omega; L_2(0,1)\|^2 \leqslant |\beta_{k,0}|^2 \|\omega; L_2(0,1)\|^2.$$

Hence, the operator $R_{0,k} = E + S_{0,k}$ is bounded $L_2(0,1) \to L_2(0,1)$ and $(R_{0,k}^{-1})^* = E - S_{0,k}^* \in L(L_2(0,1))$.
So using theorem N.K. Bary (see theorem 6.2.1 [12]) we obtain the following statement

Theorem 3. *Let $b_{1,k} \neq b_{2,k}$. Then the operator $L_{0,k}$ of problem (8), (7) have point spectrum σ_k and system $V(L_{0,k})$ of root functions in the sense of equality (13), forms a Riesz basis in $L_2(0,1)$.*

Further, we introduce operator $L_k : L_2(0,1) \to L_2(0,1)$ of the problem (6), (7).

$$L_k u \equiv L_k(D_x, z_k)u, u \in D(L_k),$$

$D(L_k) \equiv \{u \in W_2^2(0,1) : l_{1,k}u = 0, l_{2,k}u = 0\}$.

By the direct substitution we can show that the $v_{k,n}^{0,1}(x) = \sqrt{2}\sin 2\pi nx$, $v_{k,0}^{0,1} = 1 - \frac{1}{12}b_{0,k}(2x-1)$ is eigenfunctions of operator L_k: $v_{k,n}^{0,1}(x) \in D(L_k)$, $L_k v_{k,n}^{0,1}(x) = \lambda_{k,n} v_{k,n}^{0,1}(x)$, $n = 0, 1, \ldots$.

Root function of operator L_k, defined by relation

$$v_{k,n}^{1,1}(x) \equiv \sqrt{2}(1 + \beta_{k,n}(2x-1))\cos 2\pi nx + \xi_{k,n}(2x-1)^2 \sin 2\pi nx, \tag{14}$$

$$\xi_{k,n} = -b_{0,k}(8\pi n)^{-1}, n = 1, 2, ... \tag{15}$$

Hence, $V(L_k) \equiv \left\{ v_{k,n}^{s,1}(x); s = 0, 1,\ n = 0, 1, \ldots \right\}$ is the system of root functions of the operator L_k, in the sense of equality

$$L_k v_{k,n}^{1,1}(x) = \lambda_{k,n} v_{k,n}^{1,1}(x) + \rho_{k,n}^1 v_{k,n}^{1,0}(x), \tag{16}$$

$$\rho_{k,n}^1 = \xi_{k,n}^0 + \rho_{k,n}^0 \xi_{k,n},\ \rho_{k,n}^0 \equiv 8\pi n \beta_{k,n}. \tag{17}$$

Show that the system $V(L_k)$ of root functions of the operator L_k possesses a unique biorthogonal system $W(L_k)$

$$(v_{k,n}^{r,0}, w_{k,q}^{s,0}; L_2(0,1)) = \delta_{r,s}\delta_{n,q},\ r, s = 0, 1,\ q, n = 0, 1, \ldots .$$

Further, we introduce operators $R_{1,k}, S_{1,k} : L_2(0,1) \to L_2(0,1)$,

$R_{1,k} = E + S_{1,k}$, $R_{1,k} t_{k,n}^p \equiv v_{k,n}^{p,1}$, $R_{1,k} t_{k,0}^0 \equiv v_{k,1}^{0,1}$, $p = 0, 1$, $n = 0, 1$.

With formulas (16), (14) that have $S_{1,k} : L_{2,0} \to L_{2,0}$, $L_{2,1} \to 0$, that $(S_{1,k})^2 = 0$.

Show that $S_{1,k} \in L(L_2(0,1))$.

To have any function $\omega \in L_2(0,1)$,

$$\omega = \omega_0^0 t_0^0 + \sqrt{2}\sum_{m=1}^{\infty} \omega_m^0 \cos 2\pi mx + \omega_m^1 \sin 2\pi mx,\ \omega_m^j = (\omega, t_m^j; L_2(0,1)), \omega j = 0, 1,$$

$m = 0, 1, \ldots$.

$$S_{1,k}\omega = \omega_0^0 v_{k,0}^{0,1} + \sqrt{2}\sum_{m=1}^{\infty} \omega_m^0 (\beta_{k,m}(2x-1)\cos 2\pi mx + \xi_{k,m}(2x-1)^2 \sin 2\pi mx),$$

$$\|S_{1,k}\omega; L_2(0,1)\|^2 \leqslant 2(1 + |\beta_{k,n}|^2 + |\xi_{k,n}|^2)\, \|\omega; L_2(0,1)\|^2 . \tag{18}$$

So there operator $(R_{1,k}^{-1})^* = E - (S_{1,k})^*$ such that $(R_{1,k}^{-1})^* : T \to W(L_k)$, where the $W(L_k)$ is the system of functions biorthogonal to $V(L_k)$.

Of formulas (13), (14), (15) implies that the system $V(L_k)$ and $V(L_{0,k})$ a squarely close . So using theorem N.K. Bary (see theorem 6.2.3 [12]) we obtain the following statement.

Theorem 4. *Let $b_{1,k} \neq b_{2,k}$. Then the operator L_k of problem (6), (7) have point spectrum σ and system of root functions forms a Riesz basis in $L_2(0,1)$.*

The spectral problem (1), (2).

Let $b_{1,k} \neq b_{2,k}$. Then the operator L of problem (1), (2) have point spectrum

$$\sigma \equiv \left\{ \lambda_{k,n} \in \mathbb{R} : \lambda_{k,n} = 4\pi^2 n^2 + z_k^2,\ k, n = 1, 2, \ldots \right\}$$

and system of root functions

$$V(L) \equiv \left\{ v_{k,n}^s(L) \in H_1 : v_{k,n}^s(L) = v_{k,n}^{s,1}(x) v_k,\ s = 0, 1,\ k = 1, 2, \ldots, n = 0, 1, \ldots, \right\},$$

where

$$\begin{gathered} v_{k,0}(L) \equiv v_k,\ v_{k,n}^0(L) = \sqrt{2}\cos 2\pi nx v_k, \\ v_{k,n}^1(L) = \sqrt{2}(1 + b_k(2x-1))\sin 2\pi nx v_k, \end{gathered} \tag{19}$$

$n, k \in \mathbb{N}$.

System $V(L)$ of root functions of the operator L possesses a unique biorthogonal system

$$W(L) \equiv \left\{ w_{p,m}^s(L) \in H_1 : w_{p,m}^s \equiv w_{p,m}^{s,1} v_m,\ s = 0, 1,\ m = 0, 1, \ldots, p = 1, 2, \ldots \right\},$$

in the sense of equality

$$\left(v_{k,m}^j, w_{p,n}^s; H_1 \right) = \delta_{j,s}\delta_{k,p}\delta_{n,m},\ j, s = 0, 1,\ n, m = 0, 1, \ldots,\ k, p = 1, 2, \ldots .$$

Hence, we obtain the following statement.

Theorem 5. *Let* $b_{1,k} \neq b_{2,k}$*. Then the operator* L *of problem* (1), (2) *have system of root functions* $V(L)$ *complete and minimal in* H_1.

Further, we introduce operator $B \equiv (B_1 - B_2)^{-1}(B_1 + B_2) \in L(H^1)$. Then

$$\|R_{1,k}\omega; L_2(0,1)\| \leqslant C\|\omega; L_2(0,1)\|, \ \left\|(R_{1,k}^{-1})\omega; L_2(0,1)\right\| \leqslant C\left\|\omega; L_2(0,1)\right\|, C > 0.$$

So using theorem N.K. Bary (see. theorem 6.2.1, [12]), we obtain the following statement.

Theorem 6. *Let* $B \in L(H^1)$*. Then the operator* L *of problem* (1), (2) *have system of root functions* $V(L)$*, forms a Riesz basis in space* H_1.

Property problem (1), (2).

Replaced condition (2) on equivalent terms

$$\ell_3 y \equiv y(0) - y(1) + B(y(0) + y(1)) = h_3, \ \ell_2 y \equiv D_x y(0) - D_x y(1) = h_2. \tag{20}$$

Here

$$B \in L(H^1), h_3 \equiv (B_1 - B_2)^{-1}h_1 \in H^1, \ \ h_2 \in H^2.$$

Consider the particular case the problem (1), (20), if the specified $B = 0, B_0 = 0$

$$-D_x^2 y(x) + A^2 y(x) = g(x), \tag{21}$$

$$y(0) - y(1) = g_1, \ D_x y(0) - D_x y(1) = g_2 \ g_j \in H^j, \ j = 1, 2. \tag{22}$$

Theorem 7. *Let* $B = 0, B_0 = 0$*. Then for any* $g \in H_1, g_1 \in H^1, g_2 \in H^2$*, there exists a unique solution of problem* (21), (22).

Proof. We seek the solution of this problem in the form $y = u + v$, there u is the solution of the problem

$$-D_x^2 u(x) + A^2 u(x) = g(x), \ u(0) - u(1) = 0, D_x u(0) - D_x u(1) = 0, \tag{23}$$

and v is the solution of the problem

$$-D_x^2 v(x) + A^2 v(x) = 0, v(0) - v(1) = g_1, D_x v(0) - D_x v(1) = g_2. \tag{24}$$

Consider the problem (23). We expand the functions $u(x), g(x)$ in a series in the orthonormal basis in the space H_1

$$T_1 \equiv \left\{t_{k,m}^s \in H_1 : t_{k,m}^s \equiv t_m^s v_k, t_m^s \in T, v_k \in V(A)\right\},$$

$$u = \sum_{s,m,k} u_{k,m}^s t_{k,m}^s, u_{k,m}^s = (u, t_{k,m}^s; H_1),$$

$$g = \sum_{s,m,k} g_{k,m}^s t_{k,m}^s, g_{k,m}^s = (g, t_{k,m}^s; H_1).$$

Substituting into the (23) we get

$$u = \sum_{s,m,k} (4\pi^2 m^2 + {z_k}^2)^{-1} g_{k,m}^s t_{k,m}^s.$$

We estimate a number

$$-D_x^2 u = \sum_{s,m,k} (4\pi^2 m^2 + z_k^2)^{-1}(2\pi m)^2 g_{k,m}^s t_{k,m}^s, \|D_x^2 u; H_1\| \leqslant \|g; H_1\|,$$

$$A^2 u = \sum_{s,m,k} (4\pi^2 m^2 + {z_k}^2)^{-1}(z_k)^2 g_{k,m}^s t_{k,m}^s, \ \|A^2 u; H_1\| \leqslant \|g; H_1\|.$$

Hence,

$$\|u; H_2\| \leqslant \sqrt{2}\|g; H_1\|. \tag{25}$$

□

Consider the problem (24). Further, we introduce operators

$Y_j(x, A) \equiv \exp Ax + (-1)^j \exp A(1-x) \in L(H^2; H_2), j = 0, 1$. The solution of the differential equation (24) has the form

$$v(x) = Y_0(x, A)\varphi_0 + Y_1(x, A)\varphi_1, \tag{26}$$

where φ_0, φ_1 are unknown. To determine the (26) we substitute expression (26) in the condition (24) and obtain φ_0, φ_1

$$\varphi_1 = \frac{1}{2} Y_1(0, A)^{-1} g_1; \varphi_0 = \frac{1}{2} Y_1(0, A)^{-1} A^{-1} g_2.$$

Hence,

$$v = \frac{1}{2} Y_1(0, A)^{-1} g_1 + \frac{1}{2} Y_1(0, A)^{-1} A^{-1} g_2,$$

$$\|v; H_2\|^2 \leqslant C(\|g_1; H^1\|^2 + \|g_2; H^2\|^2), C > 0. \tag{27}$$

Therefore follows from inequalities (25), (27) inequality

$$\|y; H_2\|^2 \leqslant C_1(\|g; H_1\|^2 + \|g_1; H^1\|^2 + \|g_2; H^2\|^2), C_1 > 0.$$

We now return to the original problem (1), (2). Consider in connection problem as the sum $y = y_0 + y_1$, $y_j \in H_{1,j}$, $j = 0, 1$. To determine the unknowns $y_j \in H_{1,j}$ get the problem

$$-D_x^2 y_0(x) + A^2 y_0(x) = f_0(x),\ f_0(x) \in H_{1,0},$$

$$\ell_3 y_0 \equiv y_0(0) - y_0(1) = 0,\ \ell_2 y_0 \equiv D_x y_0(0) - D_x y_0(1) = h_2,$$

$$-D_x^2 y_1(x) + A^2 y_1(x) = f_1(x) - 2B_0(2x-1) y_0(x),\ f_1(x) \in H_{1,1},$$

$$y_1(0) - y_1(1) = -B(y_0(0) + y_0(1)) + h_3, D_x y_1(0) - D_x y_1(1) = 0.$$

For unknowns functions $y_j \in H_{1,j}$, $j = 0, 1$, get that problems a particular of the problem (21), (22). Hence the statement is correct

Theorem 8. *Let* $B \in L(H^1), B_0 \in L(H_2)$. *Then for any* $f \in H_1, h_1 \in H^1$, $h_2 \in H^2$, *there exists a unique solution of problem* (1), (2) *and*

$$\|y; H_2\|^2 \leqslant C_2(\|f; H_1\|^2 + \|h_1; H^1\|^2 + \|h_2; H^2\|^2),\ C_2 > 0.$$

Conclusions.

We have investigated the properties of nonlocal problem with generalized conditions Ionkin's for the Sturm-Liouville equation with polynomial potential which contains an involution operator. Defined point spectrum and built a system of root functions of the spectral problem. It is proved that under certain conditions the system of root functions spectral problem forms a Riesz basis. It is proved that under certain conditions the solution of the problem exists and only one.

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НЕЛОКАЛЬНА ГРАНИЧНА ВЕЛИЧИНА ДЛЯ АБСТРАКТНОГО ДИФЕРЕНЦIАЛЬНОГО РIВНЯННЯ ДРУГОГО ПОРЯДКУ З ОПЕРАТОРОМ IНВОЛЮЦIЇ

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РЕЗЮМЕ

Ми вивчаємо нелокальную задачу з узагальненими умовами Iонкiна для рiвняння Штурма-Лiувiлля з полiномiальним потенцiалом, що мiстить оператор iнволюцiї. Проаналiзованi спектральнi властивостi оператора цiєї задачi та встановленi умови iснування та унiкальнiсть його розв'язку. Доведено також, що система кореневих функцiй по сутi є несамосопряженным оператором аналiзованої задачi, утворює базис Рiса.

Ключовi слова: диференцiально-операторне рiвняння, коренева функцiя, оператор iнволюцiї, по сутi несамосопряженный оператор, базис Рiса, нелокальна задача.

НЕЛОКАЛЬНОЕ ГРАНИЧНОЕ ЗНАЧЕНИЕ ДЛЯ АБСТРАКТНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА С ОПЕРАТОРОМ ИНВОЛЮЦИИ

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РЕЗЮМЕ

Изучается нелокальная задача с обобщенными условиями Ионкина для уравнения Штурма-Лиувилля с полиномиальным потенциалом, который содержит оператор инволюции. Проанализированы спектральные свойства оператора этой задачи и установлены условия существования и единственности его решения. Также доказано, что система корневых функций, по существу, несамосопряженный оператор анализируемой задачи, образует базис Рисса.

Ключевые слова: дифференциально-операторное уравнение, корневая функция, оператор инволюции, по существу несамосопряженный оператор, базис Рисса, нелокальная задача.