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Belousov V.D.
Doctor of Sciences in Physics and Mathematics, Professor, Corresponding Member of the Academy of Pedagogical Sciences of the USSR

MUTUALLY INVERTIBLE QUASIGROUPS AND LOOPS

In the present paper some dependencies expressed by isotopy transformations between mutually invertible quasigroups and loops are considered. Classification of quasigroups and loops up to isotopy is given. Some propositions complementing results of A. Sade [1], S.K. Stein [15], R. Artzy [4] are proved from this point of view.
Key words: *a binary operation, a quasigroup, a loop, a right (left) invertible operation, a system of invertible quasigroups, isotopy*

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1. Let Q be some fixed finite or infinite set and A be a binary operation defined on Q. The set Q with the operation A is called a quasigroup if equations

$$A(a,x)=b, \quad A(y,a)=b$$

are uniquely solvable for arbitrary a, $b\in Q$. To be short, an operation A also will be called a *quasigroup.* If a quasigroup A has a unit, i.e., if $A(a,e)=A(e,a)=a$ then A is called a *loop.*

Two invertible operations are connected with every quasigroup A: *left invertible* operation ${}^{-1}A$ defined by the equation

$$A(y,a)=b, \quad \text{i.e.,} \quad y={}^{-1}A(b,a)$$

and *right invertible* operation A^{-1} defined by the equation

$$A(a,x)=b, \quad \text{i.e.,} \quad x=A^{-1}(a,b).$$

It is easy to see that ${}^{-1}A$ and A^{-1} are also quasigroups then for each of them there exist invertible operations; ${}^{-1}A$ has two invertible operations $({}^{-1}A)^{-1}$ and ${}^{-1}({}^{-1}A)$. But ${}^{-1}({}^{-1}A)=A$, consequently we get one new operation $({}^{-1}A)^{-1}$. In general, five invertible operations

$$A^{-1}, \quad {}^{-1}A, \quad {}^{-1}(A^{-1}), \quad ({}^{-1}A)^{-1}, \quad [{}^{-1}(A^{-1})]^{-1}$$

can be obtained. Indeed, following Stein [15] we denote some permutation of three elements a, b, c by σ: $\sigma(a,b,c)=(a',b',c')$. Then $A^{\sigma}(a,b)=c$ is equivalent to the equality $A(a',b')=c'$. For example, if $\sigma(a,b,c)=(c,a,b)$ then $A^{\sigma}={}^{-1}(A^{-1})$ because

$${}^{-1}(A^{-1})(a,b)=c \rightleftarrows A(c,a)=b.$$

If we consider all possible permutations of three elements a, b, c we obtain five invertible operations listed above and one more operation A. Note that

$$[{}^{-1}(A^{-1})]^{-1}={}^{-1}[({}^{-1}A)^{-1}]=A^*,$$

where A^* is the operation received from operation A by commutativity: $A^*(a,b)=A(b,a)$. Indeed,

$$[{}^{-1}(A^{-1})]^{-1}(a,b)=c \rightleftarrows {}^{-1}(A^{-1})(a,c)=b \rightleftarrows A^{-1}(b,c)=a \rightleftarrows A(b,a)=c,$$

i.e., $[{}^{-1}(A^{-1})]^{-1}=A^*$. Equality ${}^{-1}[({}^{-1}A)^{-1}]=A^*$ can be proved similarly.

So, a system Σ_A of six quasigroups:

$$\Sigma_A=\{A,\ {}^{-1}A,\ A^{-1},\ {}^{-1}(A^{-1}),\ ({}^{-1}A)^{-1},\ A^*\}$$

is connected with each quasigroup A which we will call a *system of invertible quasigroups for* A.

It is easy to see that, if B is one of the quasigroups of the system Σ_A then $\Sigma_B = \Sigma_A$. For example, if $B = {}^{-1}(A^{-1})$ then $B^{-1} = A^*$, ${}^{-1}B = A^{-1}$ etc.

2. Remember the definition of isotopy of two quasigroups. Let B and A be two quasigroups defined on Q. The quasigroups B and A are *isotopic* if there exist three one-to-one mappings α, β, γ of Q onto itself (permutations) such that equality

$$B(x;y) = \gamma^{-1}A(\alpha x, \beta y)$$

holds for all x, y of Q. We denote the ordered triple of the permutations (α,β,γ) by T and instead of the last equality we write briefly $B = A^T$ or $B = A^{(\alpha,\beta,\gamma)}$. If $T = (\alpha,\beta,\gamma)$ then we denote the triple $(\alpha^{-1},\beta^{-1},\gamma^{-1})$ by T^{-1}. If $S = (\alpha_1,\beta_1,\gamma_1)$ then we denote the triple $(\alpha\alpha_1,\beta\beta_1,\gamma\gamma_1)$ by TS. Then we have the following propositions:

1) *If* $B = A^T$ *then* $A = B^{T^{-1}}$.

2) *If* $C = B^T$, $B = A^S$ *then* $C = A^{ST}$, *i.e.,* $(A^S)^T = A^{ST}$.

Sometimes isotopy of quasigroups B and A will be denoted by $B \sim A$.

Propositions 1) and 2) imply that isotopy relation is symmetric and transitive and because $A = A^{(1,1,1)}$ (1 is the unitary permutation of the set Q) then $A \sim A$. Therefore isotopy relation is an equivalence on the set of all quasigroups defined on Q. The isomorphy of two quasigroups B and A is the particular case of isotopy, namely if $\alpha = \beta = \gamma$. In this case we write:

$$(\alpha,\alpha,\alpha) = \alpha \text{ and } B = A^{(\alpha,\alpha,\alpha)} = A^{\alpha}.$$

3. Consider a relationship which exists between invertible operations and isotopy. Let A^σ be an invertible quasigroup of A. By T^σ we denote the same permutation in the triple (α,β,γ) that σ permutates in the triple (a,b,c).

Lemma 1. $(A^T)^\sigma = (A^\sigma)^{T^{\sigma^{-1}}}$.

We prove the lemma only for one of the cases, for example, when $A^\sigma = {}^{-1}(A^{-1})$, i.e., $\sigma(a,b,c) = (c,a,b)$. If $T = (\alpha,\beta,\gamma)$ then $T^\sigma = (\gamma,\alpha,\beta)$. Let $(A^T)^\sigma(a,b) = c$ then $A^T(c,a) = b$ or

$$\gamma^{-1}A^\sigma(\alpha c, \beta a) = b, \qquad A^\sigma(\alpha c, \beta a) = \gamma b,$$

wherefrom

$$A^\sigma(\beta a, \gamma b) = \alpha c, \qquad \alpha^{-1}A^\sigma(\beta a, \gamma b) = c,$$

$$(A^\sigma)^{(\alpha,\beta,\gamma)}(a,b) = c, \qquad (A^\sigma)^{T^{\sigma^{-1}}}(a,b) = c,$$

consequently

$$(A^\sigma)^{T^{\sigma^{-1}}} = (A^T)^\sigma.$$

Corollary. *If* $B \sim A$ *then* $B^\sigma \sim A^\sigma$.

Considering a system of invertible quasigroups Σ_A we can notice that it is divided into subsystems of quasigroups isotopic to each other. Such a partition can be of the following types:

I) All operations of Σ_A are pairwise nonisotopic. We will name the number of equivalence classes of the set Σ_A *an index of the operation* A. Then the index of the operation A of this type equals 6.

IIa) Let $A^{-1} \sim A$. Then Corollary of Lemma 1 implies

$${}^{-1}(A^{-1}) \sim {}^{-1}A \quad \text{and} \quad [{}^{-1}(A^{-1})]^{-1} \sim ({}^{-1}A)^{-1} \quad \text{or} \quad A^* \sim ({}^{-1}A)^{-1},$$

i.e., we have the following partition:

$$\Sigma_A = \{A, A^{-1}\},\ \{{}^{-1}A,\ {}^{-1}(A^{-1})\},\ \{A^*, ({}^{-1}A)^{-1}\}.$$

Similarly we obtain two more types:

$$\text{IIb)} \quad \Sigma_A = \{A,\ {}^{-1}A\},\ \{A^{-1}, ({}^{-1}A)^{-1}\},\ \{A^*,\ {}^{-1}(A^{-1})\}.$$

$$\text{IIc)} \quad \Sigma_A = \{A, A^*\},\ \{{}^{-1}A, ({}^{-1}A)^{-1}\},\ \{A^{-1},\ {}^{-1}(A^{-1})\}.$$

The operations of the types IIa, IIb, IIc have index 3.

III) Let ${}^{-1}(A^{-1}) \sim A$ then

$${}^{-1}[{}^{-1}(A^{-1})] \sim {}^{-1}A \quad \text{or} \quad A^{-1} \sim {}^{-1}A.$$

But ${}^{-1}(A^{-1}) \sim A$ implies

$$[{}^{-1}(A^{-1})]^{-1} \sim A^{-1}, \quad \text{i.e.,} \quad A^{*} \sim A^{-1}.$$

So, in this case the operations A^{*}, A^{-1}, ${}^{-1}A$ are isotopic to each other. It remains to consider the operation $({}^{-1}A)^{-1}$. But $A^{-1} \sim {}^{-1}A$ implies

$$(A^{-1})^{-1} \sim ({}^{-1}A)^{-1}, \quad \text{i.e.,} \quad A \sim ({}^{-1}A)^{-1}.$$

Thus, the operations A, ${}^{-1}(A^{-1})$, $({}^{-1}A)^{-1}$ are isotopic to each other and we have the following partition:

$$\Sigma_A = \{A, {}^{-1}(A^{-1}), ({}^{-1}A)^{-1}\}, \quad \{A^{-1}, {}^{-1}A, A^{*}\}.$$

In this case the operation A has index 2.

IV) If $A \sim A^{-1} \sim {}^{-1}A$ then the proved statements for the types IIa, IIb imply

$$A \sim {}^{-1}(A^{-1}) \quad \text{and} \quad A \sim ({}^{-1}A)^{-1}$$

and the proved assertion for type III implies that all operations from Σ_A are isotopic to each other, i.e., the operation A has index 1. The obtained result can be formulated as the following theorem:

Theorem 1. *There exists a bijection between all types of quasigroups A and partitions of a symmetric group of degree 3 into the right cosets by its subgroups.*

Indeed, it is easy to prove that $(A^{\sigma})^{\tau} = A^{\sigma\tau}$. For this purpose we must use the definition of the invertible quasigroup A^{σ} with the help of the permutation σ of S_3. If $A^{\sigma} \sim A$ then the set of all such σ forms the subgroup H of S_3. It implies that if $A^{\sigma} \sim A^{\tau}$ then σ and τ belong to one right coset by α. Thus, there exists a subgroup of the group S_3 for every given type (and consequently some partition on the right coset of its subgroup). Below we give these correspondences for each type:

$$\text{I} \rightleftarrows \{1\}, \quad \text{IIa} \rightleftarrows \{1,(23)\}, \quad \text{IIb} \rightleftarrows \{1,(13)\},$$
$$\text{IIc} \rightleftarrows \{1,(12)\}, \quad \text{III} \rightleftarrows \{1,(123),(132)\}, \quad \text{IV} \rightleftarrows S_3.$$

Corollary 1. *The type and consequently the index of the quasigroup A are invariant under isotopy.*

The statement follows from the fact that if $B \sim A$ and $A^{\sigma} \sim A$ then $B^{\sigma} \sim B$.

The index of the operation A is defined by the order of the permutation $\sigma \neq 1$ for which $A^{\sigma} \sim A$. It implies

Corollary 2. *The index of a quasigroup is invariant under the formation of invertible operations.*

Indeed, let $B = A^{\tau}$ then $B^{\tau^{-1}} = A$, wherefrom

$$B^{\tau^{-1}\sigma} = A^{\sigma} \sim A, \quad B^{\tau^{-1}\sigma} \sim A, \quad B^{\tau^{-1}\sigma\tau} \sim A^{\tau}, \quad B^{\tau^{-1}\sigma\tau} \sim B.$$

But $\tau^{-1}\sigma\tau$ has the same order as σ.

4. Every quasigroup A can be connected with the following groups of permutations of the set Q (see [1]):

1. a group $\mathfrak{L}_A$ of the left regular permutations:

$$\mathfrak{L}_A = \{\lambda;\ \lambda A(x;y) = A(\lambda x;y),\ \forall x,y \in Q\}; \tag{1}$$

2. a group $\mathfrak{R}_A$ of the right regular permutations:

$$\mathfrak{R}_A = \{\rho;\ \rho A(x;y) = A(x;\rho y),\ \forall x,y \in Q\};$$

3. a group $\mathfrak{C}_A$ of the middle regular permutations: $\varphi \in \mathfrak{C}_A$ if there exists a permutation φ^{*}, such that $A(\varphi x;y) = A(x,\varphi^{*}y)$ for all $x,y \in Q$.

The fact that $\mathfrak{L}_A$, $\mathfrak{R}_A$, $\mathfrak{C}_A$ are groups is proved in [1]. In addition, it is possible to consider the group S^*_A consisting of all φ^*. The listed groups of permutations play the same role for quasigroups, as nuclei do for loops [5]. In [1] it is proved that if $Q(A)$ is a loop then $\mathfrak{L}_A$ is isomorphic to the left nucleus N_λ and $\mathfrak{R}_A$, $\mathfrak{C}_A$ are isomorphic to the right and the middle nucleus respectively and $\mathfrak{L}_A 1 = N_\lambda$, where

$$\mathfrak{L}_A 1 = \{\lambda 1,\ \lambda \in \mathfrak{L}_A\},$$

1 is the unit of the loop etc. If $\lambda \in \mathfrak{L}_A$ then (1) implies $A^{(\lambda,1,\lambda)} = A$. Thus, we have the particular case of autotopism of the quasigroup A. It is well known [5] that a triple of permutations $T = (\alpha, \beta, \gamma)$ of the set Q is called an autotopism of a quasigroup A if $A^T = A$. From Lemma 1 it follows that $T^{\sigma^{-1}}$ is an autotopism of the invertible quasigroup A^σ. Using this assertion it is easy to prove the following lemma:

Lemma 2. *Let A be a quasigroup then*

$$\mathfrak{L}_{^{-1}A} = \mathfrak{L}_A, \quad \mathfrak{R}_{^{-1}A} \subseteq \mathfrak{C}_A; \quad \mathfrak{L}_{A^{-1}} \subseteq \mathfrak{C}_A, \quad \mathfrak{R}_{A^{-1}} = \mathfrak{R}_A.$$

Indeed, let $\lambda \in \mathfrak{L}_A$ or $A^T = A$, where $T = (\lambda, 1, \lambda)$. The operation ^{-1}A corresponds to the permutation $\sigma = (13)$: $^{-1}A = A^{(13)}$. According to the previous remark

$$T^{\sigma^{-1}} = T^\sigma = T^{(13)} = (\lambda, 1, \lambda) = T$$

is an autotopism of the quasigroup

$$A^\sigma =^{-1} A, \quad \text{i.e.,} \quad \lambda \in \mathfrak{L}_{^{-1}A}.$$

Thus, $\mathfrak{L}_A \subseteq \mathfrak{L}_{^{-1}A}$. But this implies $\mathfrak{L}_{^{-1}A} \subseteq \mathfrak{L}_A$ because an arbitrary quasigroup can be substituted for A. Consequently $\mathfrak{L}_{^{-1}A} = \mathfrak{L}_A$. Let $\rho \in \mathfrak{R}_{^{-1}A}$. It means that $\mathfrak{C} = (1, \rho, \rho)$ is an autotopism of the quasigroup $^{-1}A = A^{(13)}$. But then $S^{(13)} = (\rho, \rho, 1)$ is an autotopism of the quasigroup A and consequently

$$A(\rho x, \rho y) = A(x, y) \quad \text{or} \quad A(\rho x, y) = A(x, \rho^{-1} y).$$

That is why, $\rho \in \mathfrak{C}_A$, i.e., $\mathfrak{R}_{^{-1}A} \subseteq \mathfrak{L}_A$. The second group of relationships of Lemma 2 can be proved in the same way.

Corollary. Let

$$\mathfrak{D}_A = \{\varphi;\ A(\varphi x, y) = A(x, \varphi^{-1} y)\}.$$

It is easy to see that $\mathfrak{D}_A$ is a group. We call it the group of central regular permutations. Obviously $\mathfrak{D}_A \subseteq \mathfrak{L}_A \cap \mathfrak{L}^*_A$. $\mathfrak{R}_{^{-1}A} \subseteq \mathfrak{D}_A$ arises from the foregoing. Reasoning in the invertible way, it easy to prove the following

$$\mathfrak{D}_A \subseteq \mathfrak{R}_{^{-1}A}, \quad \text{i.e.,} \quad \mathfrak{D}_A = \mathfrak{R}_{^{-1}A}.$$

Similarly we obtain $\mathfrak{L}_{A^{-1}} = \mathfrak{D}_A$.

Lemma 3.
1) If A has the left unit, then $\mathfrak{R}_A = \mathfrak{L}_{^{-1}A}$.
3) If A has the right unit, then $\mathfrak{L}_A = \mathfrak{C}_{A^{-1}}$.

Let $\varphi \in \mathfrak{C}_{^{-1}A}$, it means that there exists a permutation φ^* such that

$$^{-1}A(\varphi x, y) =^{-1} A(x, \varphi^* y)$$

for all $x, y \in Q$.

Let $^{-1}A(\varphi x, y) = z$ then $\varphi x = A(z, y)$. Simultaneously we have $x = A(z, \varphi^* y)$ then $\varphi A(z, \varphi^* y) = A(z, y)$. In this equality z and y can be any elements, in particular, let $z = 1$ then $\varphi\varphi^* y = y$ or $\varphi^* = \varphi^{-1}$. That is why we have

$$\varphi A(z, \varphi^{-1} y) = A(z, y) \quad \text{or} \quad \varphi A(z, y) = A(z, \varphi y), \quad \text{,i.e.,} \quad \varphi \in \mathfrak{R}_A$$

consequently $\mathfrak{C}_{^{-1}A} \subseteq \mathfrak{R}_A$. From Lemma 2 $\mathfrak{R}_A \subseteq \mathfrak{C}_{^{-1}A}$ then $\mathfrak{R}_A = \mathfrak{C}_{^{-1}A}$. The second statement of Lemma 3 can be proved similarly.

Remark. If A is a loop (with a unit 1) then: 1) A^* is a loop; 2) A^{-1} and $^{-1}(A^{-1})$ have the left unit 1; 3) ^{-1}A and $(^{-1}A)^{-1}$ have the right unit 1.

Theorem 2. *Let A be a quasigroup. Then the following relations hold for groups of regular permutations of invertible operations:*

$$\begin{array}{llllllll}
1) & \mathfrak{L}_{^{-1}A} = \mathfrak{L}_A & 2) & \mathfrak{L}_{^{-1}A} = \mathfrak{D}_A & 3) & \mathfrak{L}_{A^*} = \mathfrak{R}_A & 4) & \mathfrak{L}_{(^{-1}A)^{-1}} = \mathfrak{R}_A \\
 & \mathfrak{L}_{^{-1}A} = \mathfrak{L}_A & & \mathfrak{L}_{^{-1}A} = \mathfrak{L}_A & & \mathfrak{L}_{^{-1}A} = \mathfrak{L}_A & & \mathfrak{L}_{^{-1}A} = \mathfrak{L}_A \\
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5) & \mathfrak{L}_{^{-1}(A^{-1})} = \mathfrak{D}_A & & & & & & \\
 & \mathfrak{R}_{^{-1}(A^{-1})} = \mathfrak{L}_A & & & & & & \\
 & \mathfrak{D}_{^{-1}(A^{-1})} = \mathfrak{R}_A & & & & & &
\end{array}$$

If A is a loop then the relations

$$\mathfrak{C}_{^{-1}A} = \mathfrak{C}_{^{-1}(A^{-1})} = \mathfrak{L}_A, \quad \mathfrak{C}_{A^{-1}} = \mathfrak{C}_{^{-1}(A^{-1})} = \mathfrak{R}_A, \quad \mathfrak{C}_{A^*} = \mathfrak{C}^*_A$$

are added.

``Proof.''}We prove the equalities of the group 1). Lemma 2 and its corollaries imply the first and the second equalities. Since A is an arbitrary operation then replacing A with ${}^{-1}A$ we obtain

$$\mathfrak{R}_{^{-1}(^{-1}A)} = \mathfrak{D}_{^{-1}A}, \quad \text{i.e.,} \quad \mathfrak{D}_{^{-1}A} = \mathfrak{R}_A.$$

The second group of the equalities is proved in the same way. The rest of the equalities follow from the equalities of 1) and 2).

For example, $\mathfrak{R}_{^{-1}(A^{-1})} = \mathfrak{D}_{A^{-1}} = \mathfrak{L}_A$.

Relationships for the groups of the middle regular permutations are proved on Lemma 3 and on five groups of equalities of Theorem 2.

5. If A is a loop then Σ_A does not consist only of loops. So, if $A(x,y) = xy$ is a group then $A^{-1}(x,y) = x^{-1}y$ and, in general, A^{-1} has no a right unit. That is why it is reasonable to consider such isotopes of invertible operations for A which are also loops. We will do it in the following way. If A is a loop with 1 then we consider a permutation I defined by $A(x, Ix) = 1$, i.e., Ix is a right invertible of x.

We introduce the following two operations:

$$A^\rho(x,y) = A^{-1}(I^{-1}x, y), \quad A^\lambda(x,y) = {}^{-1}A(x, Iy).$$

We show that A^ρ and A^λ are loops with the same unit 1. The definition of 1 implies $I1 = 1$. We have seen above, if A is a loop then A^{-1} has the left unit 1. That is why

$$A^\rho(1,y) = A^{-1}(I^{-1}1, y) = A^{-1}(1,y) = y.$$

Now let $A^\rho(x,1) = x'$. Using the definition of the operation A^ρ we conclude that $A(I^{-1}x, x') = 1$, but $A(x, Ix) = 1$ implies $A(I^{-1}x, x) = 1$, wherefrom $x = x'$. Two loops A^ρ and A^λ will be called a right loop and a left loop for A respectively. Thus, we can associate the loops A^ρ and A^λ with every loop A. Their consideration is prompted by invertible operations for groups or more generally IP-loop [5]. The definitions of A^ρ and A^λ can be rewritten in the following way:

$$A^\rho = (A^{-1})^{P_A}, \quad A^\lambda = ({}^{-1}A)^{S_A},$$

where $P_A = (I^{-1}, 1, 1)$, $S_A = (1, I, 1)$ and $I = I_A$ are defined by the equality $A(x, I_A x) = 1$. It is easy to show that $I_{A^\rho} = I_{A^\lambda} = I^{-1}$. Hence as a result we obtain relationships:

$$P_{A^\rho} = P_{A^\lambda} = P_A^{-1}, \quad S_{A^\rho} = S_{A^\lambda} = S_A^{-1}.$$

As for the invertible quasigroups we can consider invertible loops for loops A^ρ and A^λ etc., and we define: $(A^\rho)^\rho = A^{\rho^2}$, $(A^\rho)^\lambda = A^{\rho\lambda}$ etc.

For quasigroups we have seen that there exist only six invertible quasigroups (including the quasigroup itself). To clarify analogical question for invertible loops, we give the following lemmas without proof:

Lemma 4. $A^{\rho^2} = A$, $A^{\lambda^2} = A$.

Lemma 5.

$$\begin{cases} A^{\rho\lambda} = [{}^{-1}(A^{-1})]^{(1, I^{-1}, I^{-1})}, \\ A^{\rho\lambda} = [{}^{-1}(A^{-1})]^{(I,1,I)}; \end{cases} \qquad \begin{cases} A^{\rho\lambda\rho} = (A^*)^{I^{-1}}, \\ A^{\lambda\rho\lambda} = (A^*)^{I}. \end{cases}$$

Thus, two previous lemmas imply that the system Σ_A^0 of invertible loops of a loop A consists of the operations:

$$A,\ A^{\rho},\ A^{\lambda},\ A^{\rho\lambda},\ A^{\lambda\rho},\ A^{\rho\lambda\rho},\ A^{\lambda\rho\lambda},\ A^{\rho\lambda\rho\lambda},\ A^{\lambda\rho\lambda\rho},\ldots.$$

But Lemma 4 and Lemma 5 and the following lemma imply that there are only six loops with the precision up to isomorphism.

Lemma 6. $A^{\rho\alpha}=A^{\alpha\rho}$, $A^{\lambda\alpha}=A^{\alpha\lambda}$, *where* α *is an arbitrary permutation of the set* Q.

Now consider the last two equalities of Lemma 5. From them we deduce the equality $A^{\rho\lambda\rho I}=A^{\lambda\rho\lambda I^{-1}}$, wherefrom $A^{\lambda\rho\lambda}=(A^{\rho\lambda\rho})^{I^2}$, i.e., $A^{\lambda\rho\lambda}$ and $A^{\rho\lambda\rho}$ are isomorphic. That is why in the system Σ_A^0 all loops of the form $A^{\lambda\rho\lambda\rho}\ldots$ or $A^{\rho\lambda\rho\lambda}\ldots$ having more than three letters in the upper index are isomorphic to the following six loops:

$$A,\ A^{\rho},\ A^{\lambda},\ A^{\rho\lambda},\ A^{\lambda\rho},\ A^{\rho\lambda\rho}.$$

6. We call a loop A a G-*loop* if it has the following property (G): if a loop B is isotopic to A then B is isomorphic to A. It is well known [6] that groups are G-loops, some Moufang loops are also G-loops. The following theorem shows that the property (G) is invariant under the formation of invertible loops.

Theorem 3. *If* A *is a* G*-loop then every operation* A^{σ} *of* Σ_A^0 *is a* G*-loop.*

It is enough to consider the principal isotopes (B is principal isotope of A if $B=A^{(\alpha,\beta,1)}$) because any isotope of A is isomorphic to a principal isotopy [5]. If a loop B is principally isotopic to A: $B=A^{T}$ then the isotopism T has a form: $T=(R_k^{-1},L_\ell^{-1},1)$ where $R_kx=A(x,k)$, $L_\ell x=A(\ell,x)$ [5]. In this case $m=A(\ell,k)$ is a unit of the loop B. Let us formulate the lemma which we will accept without proof:

Lemma 7. $(A^T)^{\rho}=(A^{\rho})^{T^{\rho}}$, $(A^T)^{\lambda}=(A^{\lambda})^{T^{\lambda}}$, *where* $T^{\rho}=(IV_m^{-1}L_\ell^{-1},1,V_aL_\ell^{-1})$, $T^{\lambda}=(1,I^{-1}V_mR_k^{-1},R_k^{-1})$ *and the permutation* V_a *is determined by the equality* $V_ax=A^{-1}(x,a)$.

``Proofof Theorem 3." It is enough to show that A^{ρ} and A^{λ} are G-loops and to the other invertible loops, the induction is applicable. If A is G-loop then for any elements k and $l\in Q$ there exist a permutation α of Q such that $A^T=A^{\alpha}$, $T=(R_k^{-1},L_\ell^{-1},1)$. Prove that A^{ρ} is also a G-loop. For this purpose we calculate $(A^T)^{\rho}$. Since $A^T=A^{\alpha}$ then in view of Lemma 6 and Lemma 7 $(A^T)^{\rho}=A^{\alpha\rho}$ or $(A^{\rho})^{T\rho}=A^{\rho\alpha}$. We rewrite the last equation in a more detail way:

$$(A^{\rho})^{(IV_m^{-1}L_\ell^{-1},\,1,\,L_\ell^{-1})}=A^{\rho\alpha},$$

wherefrom

$$(A^{\rho})^{(IV_m^{-1},\,L_\ell,\,1)}=(A^{\rho})^{\alpha L_\ell}. \tag{2}$$

Let $\tilde R_ux=A^{\rho}(x,u)$, $\tilde L_vx=A^{\rho}(v,x)$. Calculating these two permutations. We have

$$\tilde R_ux=A^{-1}(I^{-1}x,u)=V_uI^{-1}x,$$

wherefrom $\tilde R_u=V_uI^{-1}$. For $\tilde L_v$ we have $\tilde L_vx=A^{-1}(I^{-1}v,x)$, wherefrom

$$A(I^{-1}v,\tilde L_vx)=x,\quad L_{I^{-1}v}\tilde L_v=1$$

consequently $\tilde L_v=L^{-1}_{I^{-1}v}$. From the equalities for $\tilde R_u$ and $\tilde L_v$ we find

$$IV_u^{-1}=\tilde R_u^{-1},\quad L_{I^{-1}v}=\tilde L_v^{-1}\quad\text{or}\quad \tilde L_v=L^{-1}_{I^{-1}v},$$

therefore the equality (2) has the form

$$(A^{\rho})^{(\tilde R_m^{-1},\tilde L_{I\ell}^{-1},1)}=(A^{\rho})^{\alpha L_\ell}.$$

But $m=A(\ell,k)$ and if $I\ell=n$ then $\ell=I^{-1}n$, $k=A^{-1}(I^{-1}n,m)$, wherefrom

$$(A^{\rho})^{(\tilde R_m^{-1},\tilde L_n^{-1},1)}=(A^{\rho})^{\alpha L_\ell}. \tag{3}$$

Since m and n can be arbitrary elements from Q then (3) shows that a loop being principally isotopic to the loop A^{ρ} is isomorphic to A^{ρ}, i.e., A^{ρ} is also a G-loop. The fact that A^{λ} is a G-loop can be proved analogically.

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ВЗАЄМНО-ОБОРОТНІ КВАЗІГРУПИ ТА ЛУПИ

Білоусов В.Д.

РЕЗЮМЕ

В цій статті розглядаються деякі залежності між взаємно-оборотними квазігрупами і лупами за допомогою перетворення ізотопії, дана класифікація квазігруп і луп за класами і з цієї точки зору доводиться ряд тверджень, які доповнюють результати А. Сада [1], Ш. Стейна [15], Р. Арці [4].

Ключові слова: бінарна операція, квазігрупа, лупа, права (ліва) оборотна операція, система оборотних квазігруп, ізотоп.

ВЗАИМООБРАТНЫЕ КВАЗИГРУППЫ И ЛУПЫ

Белоусов В.Д.

РЕЗЮМЕ

В настоящей статье рассматриваются некоторые зависимости между взаимообратными квазигруппами и лупами с помощью преобразования изотопии, дается классификация квазигрупп и луп по родам с этой точки зрения и доказывается ряд предложений дополняющих результаты А. Сада [1], Ш. Стейна [15], Р. Арци [4].

Ключевые слова: бинарная операция, квазигруппа, лупа, правая (левая) обратная операция, система обратимых квазигрупп, изотоп.