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THE DEGREE OF THE ALGEBRA OF JOINT INVARIANTS OF TWO BINARY FORMS

We calculate the degree of the algebra of joint invariants $\mathcal{I}_{d_1,d_2}$ of two binary forms. For this purpose we used the explicit formula for the Poincaré series $\mathcal{P}(\mathcal{C}_d, z)$ of this algebra.
Keywords: *invariant, binary form, Poincaré series, algebra of joint invariants, degree of algebra*

Introduction

Let $R = R_0 \oplus R_1 \oplus \cdots$ be a finitely generated graded complex algebra, $R_0 = \mathbb{C}$. Denote by

$$\mathcal{P}(R, z) = \sum_{j=0}^{\infty} \dim R_j z^j,$$

its Poincaré series. Letting r be the transcendence degree of the quotient field of R over $\mathbb{C}$, the number

$$\deg(R) := \lim_{z \to 1} (1-z)^r \mathcal{P}(R, z),$$

is called the degree of the algebra R. The first two terms of the Laurent series expansion of $\mathcal{P}(R, z)$ at the point $z = 1$ have the following form

$$\mathcal{P}(R, z) = \frac{\deg(R)}{(1-z)^r} + \frac{\psi(R)}{(1-z)^{r-1}} + \cdots$$

The numbers $\deg(R), \psi(R)$ are important characteristics of the algebra R. For instance, if R is an algebra of invariants of a finite group G then $\deg(R)^{-1}$ is order of the group G and $2\dfrac{\psi(R)}{\deg(R)}$ is the number of pseudo-reflections in G, see [4].

Let V_d be the standard $(d+1)$-dimensional complex representation of SL_2 and let $\mathcal{I}_d := \mathbb{C}[V_d]^{SL_2}$ be the corresponding algebra of invariants. In the language of classical invariant theory the algebra $\mathcal{I}_d$ is called the algebra of invariants for binary forms of degree d. The following explicit formula for the degree $\deg(\mathcal{I}_d)$ was derived by Hilbert in [5]:

$$\deg(\mathcal{I}_d) = \begin{cases} -\dfrac{1}{4d!} \displaystyle\sum_{0 \leqslant e < d/2} (-1)^e \binom{d}{e} \left(\frac{d}{2} - e\right)^{d-3}, & \text{if } d \text{ is odd,} \\ -\dfrac{1}{2d!} \displaystyle\sum_{0 \leqslant e < d/2} (-1)^e \binom{d}{e} \left(\frac{d}{2} - e\right)^{d-3}, & \text{if } d \text{ is even.} \end{cases}$$

In [8] and [9] Springer obtained two different proofs of this result. Also, he found an integral representation and the asymptotic behavior for Hilbert's constants. For this purpose Springer [8] derived an explicit formula for the Poincaré series $\mathcal{P}(\mathcal{I}_d, z)$. V.L.Popov calculated $\psi(\mathcal{I}_d)$ in [7].

Let $\mathcal{C}_d$ be the algebra of the covariants of binary d-forms, i.e. $\mathcal{C}_d \cong \mathbb{C}[V_1 \oplus V_d]^{SL_2}$.

We calculated $\deg(\mathcal{C}_d)$ and $\psi(\mathcal{C}_d)$ in [3]. For this purpose we used the explicit formula for the Poincaré series $\mathcal{P}(\mathcal{C}_d, z)$ derived by L.Bedratyuk in [1]. Also, we calculated both an integral representation and the asymptotic behavior of the constants.

Let V_{d_1}, V_{d_2} be the vector $\mathbb{C}$-spaces of the binary forms of degrees d_1 and d_2 endowed with the natural action of the group $SL_2(\mathbb{C})$. Consider the induced action of the group $SL_2(\mathbb{C})$ on the algebras of the polynomial functions $\mathcal{O}(V_{d_1} \oplus V_{d_2})$ and $\mathcal{O}(V_{d_1} \oplus V_{d_2} \oplus \mathbb{C}^2)$. The algebras

$$\mathcal{I}_{d_1,d_2} := \mathcal{O}(V_{d_1} \oplus V_{d_2})^{SL_2(\mathbb{C})} \text{ and } \mathcal{C}_{d_1,d_2} := \mathcal{O}(V_{d_1} \oplus V_{d_2} \oplus \mathbb{C}^2)^{SL_2(\mathbb{C})},$$

are called the algebra of joint invariants and the algebra of joint covariants for the binary forms. We found $\deg(\mathcal{C}_{d_1,d_2})$, using the explicit formula for the Poincaré series $\mathcal{P}(\mathcal{C}_{d_1,d_2}, z)$ in [10].

In the present paper, acting in the spirit of Springer's papers, we calculate $\deg(\mathcal{I}_{d_1,d_2})$. We consider 2 cases. Case 1: $d_2 - d_1 = 1 \pmod 2$. Case 2: $d_1 = d_2 \pmod 2$.

Computation $\deg(\mathcal{I}_{d_1,d_2})$ **if** d_1 **and** d_2 **have different parity**

The reductivity of $SL_2(\mathbb{C})$ implies that the algebra $\mathcal{I}_{d_1,d_2}$ is finitely generated $\mathbb{Z}$-graded algebra

$$\mathcal{I}_{d_1,d_2} = (\mathcal{I}_{d_1,d_2})_0 + (\mathcal{I}_{d_1,d_2})_1 + \cdots + (\mathcal{I}_{d_1,d_2})_i + \cdots,$$

where each of subspaces $(\mathcal{I}_{d_1,d_2})_i$, of joint invariants of degree i is finite dimensional.

The formal power series $\mathcal{P}\mathcal{I}_{d_1,d_2}(z), \in \mathbb{Z}[[z]]$,

$$\mathcal{P}\mathcal{I}_{d_1,d_2}(z) = \sum_{i=0}^{\infty} \dim((\mathcal{I}_{d_1,d_2})_i) z^i$$

are called the Poincaré series of the algebras of joint invariants and covariants. The finitely generation of the algebras $\mathcal{I}_{d_1,d_2}$, and $\mathcal{C}_{d_1,d_2}$ implies that their Poincaré series are expansions of certain rational functions.

The following theorem shows an explicit form for these rational functions in the case $d_2 - d_1 = 1 \pmod 2$.

Let φ_n, $n \in \mathbb{N}$ be the linear operator that transforms a rational function f in z to a rational function $\varphi_n(f)$ which is defined on the power z^n by

$$(\varphi_n(f))(z^n) = \frac{1}{n} \sum_{j=0}^{n-1} f(\zeta_n^j z), \qquad \zeta_n = e^{\frac{2\pi i}{n}}.$$

Theorem 1. *(see [2]) Let* $d_2 - d_1 = 1 \pmod 2$ *and* $d_2 > d_1$. *Then the Poincaré series* $\mathcal{P}(\mathcal{I}_{d_1,d_2}, z)$ *and* $\mathcal{P}(\mathcal{C}_{d_1,d_2}, z)$ *have the following form*

$$\mathcal{P}(\mathcal{I}_{d_1,d_2}, z) = \sum_{d_1/2 \leqslant k \leqslant d_1} \varphi_{2k-d_1}\left((1-z^2)A_k(z)\right) + \sum_{k=0}^{[d_2/2]} \varphi_{d_2-2k}\left((1-z^2)B_k(z)\right),$$

where

$$A_k(z) = \frac{(-1)^{\frac{1}{2}(d_2-d_1+1)} z^{(d_1-k)(d_1-k+1)+\frac{1}{4}(d_1+d_2-2k+1)^2}}{(z^2,z^2)_k (z^2,z^2)_{d_1-k} (z,z^2)_{\frac{d_2+d_1+1}{2}-k} (z,z^2)_{\frac{d_2-d_1+1}{2}+k}},$$

$$B_k(z) = \begin{cases} \dfrac{(-1)^k z^{k(k+1)}}{(z^{d_2-d_1-2k},z^2)_{d_1+1}(z^2,z^2)_k (z^2,z^2)_{d_2-k}} & \text{for } 2k < d_2 - d_1, \\[2ex] \dfrac{(-1)^{\frac{d_2-d_1-1}{2}} z^{k(k+1)+1/4(d_2-d_1-1-2k)^2}}{(z,z^2)_{s+1}(z,z^2)_{d_1-s}(z^2,z^2)_k (z^2,z^2)_{d_2-k}} & \text{for } s = \dfrac{2k-(d_2-d_1)-1}{2}. \end{cases}$$

here $(a,q)_n = (1-a)(1-a\,q)\cdots(1-a\,q^{n-1})$ *denotes the* q*-shifted factorial*

In order to calculate the rational coefficient $\deg(\mathcal{I}_d)$ we shall use several auxiliary facts.

Lemma 1. *(see [10])*

(*i*) *The Laurent series for* $(1+z)A_k(z)$ *at* $z=1$ *starts off with*

$$\frac{1}{(1-z)^{d_2+d_1+1}} \cdot \frac{(-1)^{\frac{1}{2}(d_2-d_1+1)}}{2^{d_1-1} k!(d_1-k)!(d_2+d_1-2k)!!(d_2-d_1+2k)!!} -$$
$$- \frac{1}{(1-z)^{d_2+d_1}} \cdot \frac{(-1)^{\frac{1}{2}(d_2-d_1+1)}\left((d_1-2k+1)(d_1+d_2+2)\right)}{2^{d_1} k!(d_1-k)!(d_2+d_1-2k)!!(d_2-d_1+2k)!!} + \cdots$$

(*ii*) *The Laurent series for* $(1+z)B_k(z)$ *at* $z=1$, *starts off with*

$$\frac{1}{(1-z)^{d_1+d_2+1}} \cdot \frac{(-1)^k}{2^{d_2-1} k!(d_2-k)! \prod_{i=0}^{d_1}(d_2-d_1-2k+2i)} +$$
$$+ \frac{(-1)^k}{(1-z)^{d_2+d_1}} \cdot \frac{(d_1+d_2+2)(d_2-2k-1)}{2^{d_2} k!(d_2-k)! \prod_{i=0}^{d_1}(d_2-d_1-2k+2i)} + \cdots$$

The following lemma shows how the function φ_n acts on the negative powers of $1-z$.

Lemma 2. *(see[3,10])*

For $h \in \mathbb{N}$

(i) $$\varphi_n\left(\frac{1}{(1-z)^h}\right) = \sum_{i=0}^{h} \frac{\alpha_{n\,i}}{(1-z)^i},$$

where $\alpha_{nh} = n^{h-1}$ *and* $\alpha_{n,h-1} = -n^{h-2}(n-1)\dfrac{h}{2}$,

(ii) $$z\varphi_n\left(\frac{1}{(1-z)^h}\right) = \sum_{i=0}^{h} \frac{\beta_{n\,i}}{(1-z)^i}, h \in \mathbb{N},$$

where $\beta_{nh} = n^{h-1}, \beta_{n,h-1} = -n^{h-2}\left((n-1)\dfrac{h}{2} + n\right)$.

The following lemma follows from lemma $2(i)$.

Lemma 3. *(see [8]) Let* $f \in \mathbb{C}(z)$. *The poles of* $\varphi_n(f)$ *are of the form* α^n, *where* $\alpha \neq 0$ *is a pole of* f. *If there is only one pole* α *of* f *of maximal order* h, *then there is only one pole of maximal order of* $\varphi_n(f)$, *viz.* α^n, *and this order is also* h.

Since

$$GCD(\{d_i - 2k\}) = \begin{cases} 1, \text{ if } d_i \text{ is odd }, \\ 2, \text{ if } d_i \text{ is even} \end{cases} \quad (i = 1, 2; \quad 1 \leqslant k < d_i/2)$$

it follows that in case $d_2 - d_1 = 1 \pmod 2$ the function $\mathcal{P}(\mathcal{I}_{d_1,d_2}, z)$ has only one pole of maximal order that is $z = 1$ (by the Theorem 1).

Let's calculate the coefficient $\deg(\mathcal{I}_{d_1,d_2})$.

Theorem 2.

$$\deg(\mathcal{I}_{d_1,d_2}) = -(g(d_1, d_2, d_1) + g(d_2, d_1, d_2)),$$

where $g(d,b,c) = \frac{1}{4d!}\sum_{k=0}^{c/2} \dfrac{(-1)^k \binom{d}{k}(\frac{d}{2} - k)^{d+b-2}}{\prod_{i=0}^{b}(\frac{d-b}{2} - k + i)}$.

Proof Combining Lemma1 and Lemma2, we obtain

$$\sum_{d_1/2 \leqslant k \leqslant d_1} \varphi_{2k-d_1}\left((1-z^2)A_k(z)\right) =$$
$$= \frac{1}{(1-z)^{d_2+d_1}} \sum_{d_1/2 \leqslant k \leqslant d_1} \frac{(-1)^{\frac{1}{2}(d_2-d_1-1)}(2k-d_1)^{d_2+d_1-1}}{2^{d_1-1}k!(d_1-k)!(d_2+d_1-2k)!!(d_2-d_1+2k)!!} +$$
$$+ \frac{1}{(1-z)^{d_2+d_1-1}} \sum_{d_1/2 \leqslant k \leqslant d_1} \frac{(-1)^{\frac{1}{2}(d_2-d_1-1)}(2k-d_1)^{d_2+d_1-2}(2k-d_1-1)}{2^{d_1-1}k!(d_1-k)!(d_2+d_1-2k)!!(d_2-d_1+2k)!!} \cdots.$$

In the same way, we obtain

$$\sum_{0 \leqslant k < [d_2/2]} \varphi_{d_2-2k}\left((1-z^2)B_k(z)\right) =$$
$$= \frac{1}{(1-z)^{d_2+d_1}} \sum_{0 \leqslant k \leqslant d_2/2} \frac{(-1)^k (d_2-2k)^{d_2+d_1-1}}{2^{d_2-1}k!(d_2-k)!\prod_{i=0}^{d_1}(d_2-d_1-2k+2i)} +$$
$$+ \frac{1}{(1-z)^{d_2+d_1-1}} \sum_{0 \leqslant k \leqslant d_2/2} \frac{(-1)^k (d_2-2k)^{d_2+d_1-2}(d_2-2k-1)}{2^{d_2-1}k!(d_2-k)!\prod_{i=0}^{d_1}(d_2-d_1-2k+2i)} + \cdots.$$

Taking into account Theorem 1, we get

$$\mathcal{P}(\mathcal{I}_{d_1,d_2}, z) = \frac{1}{(1-z)^{d_2+d_1}}\Big(\frac{1}{d_1!2^{d_1-1}} \sum_{d_1/2 \leqslant k \leqslant d_1} \binom{d_1}{k} \frac{(-1)^{\frac{1}{2}(d_2-d_1-1)}(2k-d_1)^{d_2+d_1-1}}{(d_2+d_1-2k)!!(d_2-d_1+2k)!!} +$$
$$+\frac{1}{d_2!2^{d_2-1}} \sum_{0 \leqslant k \leqslant d_2/2} \binom{d_2}{k} \frac{(-1)^k (d_2-2k)^{d_2+d_1-1}}{\prod_{i=0}^{d_1}(d_2-d_1-2k+2i)}\Big)+$$
$$+\frac{1}{(1-z)^{d_2+d_1-1}}\Big(\frac{1}{d_1!2^{d_1-1}} \sum_{d_1/2 \leqslant k \leqslant d_1} \binom{d_1}{k} \frac{(-1)^{\frac{1}{2}(d_2-d_1-1)}(2k-d_1)^{d_2+d_1-1}}{(d_2+d_1-2k)!!(d_2-d_1+2k)!!} +$$
$$+\frac{1}{d_2!2^{d_2-1}} \sum_{0 \leqslant k \leqslant d_2/2} \binom{d_2}{k} \frac{(-1)^k (d_2-2k)^{d_2+d_1-1}}{\prod_{i=0}^{d_1}(d_2-d_1-2k+2i)}-$$
$$-\frac{1}{d_1!2^{d_1-1}} \sum_{d_1/2 \leqslant k \leqslant d_1} \binom{d_1}{k} \frac{(-1)^{\frac{1}{2}(d_2-d_1-1)}(2k-d_1)^{d_2+d_1-1}}{(d_2+d_1-2k)!!(d_2-d_1+2k)!!} -$$
$$-\frac{1}{d_2!2^{d_2-1}} \sum_{0 \leqslant k \leqslant d_2/2} \binom{d_2}{k} \frac{(-1)^k (d_2-2k)^{d_2+d_1-2}}{\prod_{i=0}^{d_1}(d_2-d_1-2k+2i)}\Big).$$

By Luna's Slice Theorem, see [6], we have $\operatorname{tr}\deg_{\mathbb{C}} \mathcal{I}_{d_1,d_2} = d_1 + d_2 - 1$. It now follows that

$$\frac{1}{d_1!2^{d_1-1}} \sum_{d_1/2 \leqslant k \leqslant d_1} \binom{d_1}{k} \frac{(-1)^{\frac{1}{2}(d_2-d_1-1)}(2k-d_1)^{d_2+d_1-1}}{(d_2+d_1-2k)!!(d_2-d_1+2k)!!} +$$
$$+\frac{1}{d_2!2^{d_2-1}} \sum_{0 \leqslant k \leqslant d_2/2} \binom{d_2}{k} \frac{(-1)^k (d_2-2k)^{d_2+d_1-1}}{\prod_{i=0}^{d_1}(d_2-d_1-2k+2i)} = 0.$$

Thus

$$\deg(\mathcal{I}_{d_1,d_2}) = -\frac{1}{2^{d_1-1}d_1!} \sum_{k=[\frac{d_1}{2}]+1}^{d_1} \binom{d_1}{k} \frac{(-1)^{\frac{d_2-d_1+1}{2}}(2k-d_1)^{d_2+d_1-2}}{(d_2+d_1-2k)!!(d_2-d_1+2k)!!} -$$
$$-\frac{1}{2^{d_2-1}d_2!} \sum_{k=0}^{\frac{d_2}{2}} \binom{d_2}{k} \frac{(-1)^k (d_2-2k)^{d_2+d_1-2}}{\prod_{i=0}^{d_1}(d_2-d_1-2k+2i)}.$$

This completes the proof of theorem. □

Computation $\deg(\mathcal{I}_{d_1,d_2})$ if d_1 and d_2 have same parity.

We use the Poincaré series for the algebra of joint invariants of two binary forms to calculate the degree of this algebra.

Theorem 3. *(see [2,10]) For $d_1 = d_2 \pmod 2$ and $d_2 > d_1$ the Poincaré series $\mathcal{PI}_{d_1,d_2}(z)$ are calculated by the formula*

$$\mathcal{PI}_{d_1,d_2}(z) = \sum_{d_1/2 \leqslant k \leqslant d_1} \varphi_{2k-d_1}\left((1-z^2)A_k(z)\right) +$$
$$+\sum\nolimits_{d_1/2 \leqslant k \leqslant d_1}\left(z\,\varphi_{2k-d_1}\left((1-z^2)B_k(z)\right)\right)'_z + \sum_{0 \leqslant 2k \leqslant d_2-d_1-2} \varphi_{d_2-2k}\left((1-z^2)C_k(z)\right),$$

where

$$A_k(z) = (-1)^{\frac{d_1-d_2}{2}} z^{(1+d_1-k)(d_1-k)+(1+\frac{d_1+d_2}{2}-k)(\frac{d_1+d_2}{2}-k)} \times$$
$$\times \frac{2k - \frac{d_1-d_2}{2} - \sum_{i=d_1-k+1}^{k}\left(\frac{1}{1-z^{2i}} + \frac{1}{1-z^{2i-d_1+d_2}}\right)}{(z^2,z^2)_k (z^2,z^2)_{d_1-k} (z^2,z^2)_{k-\frac{d_1-d_2}{2}} (z^2,z^2)_{\frac{d_1+d_2}{2}-k}}$$

$$B_k(z) = \frac{(-1)^{\frac{d_1-d_2}{2}} z^{(1+d_1-k)(d_1-k)+(1+\frac{d_1+d_2}{2}-k)(\frac{d_1+d_2}{2}-k)}}{(z^2,z^2)_k (z^2,z^2)_{d_1-k} (z^2,z^2)_{\frac{d_2}{2}+k-\frac{d_1}{2}} (z^2,z^2)_{\frac{d_2}{2}-(k-\frac{d_1}{2})}}$$

$$C_k(z) = \frac{(-1)^k z^{k(k+1)}}{(z^{d_2-d_1-2k},z^2)_{d_1+1} (z^2,z^2)_k (z^2,z^2)_{d_2-k}}.$$

We need the following facts to calculate the rational coefficient $\deg(\mathcal{I}_d)$.

Lemma 4. *(see [10]) The following statements hold*

(i) *The first terms of the Laurent series for $(1+z)B_k(z)$ at $z=1$ are*

$$\frac{(-1)^{\frac{d_1-d_2}{2}}}{(1-z)^{d_1+d_2}}\cdot\frac{1}{2^{d_1+d_2-1}k!(d_1-k)!\left(\frac{d_2-d_1}{2}+k\right)!\left(\frac{d_2+d_1}{2}-k\right)!}+$$
$$+\frac{(-1)^{\frac{d_1-d_2}{2}}}{(1-z)^{d_1+d_2-1}}\cdot\frac{(d_1+d_2+2)(2k-d_1-1)+1}{2^{d_1+d_2}k!(d_1-k)!\left(\frac{d_2-d_1}{2}+k\right)!\left(\frac{d_2+d_1}{2}-k\right)!}+\dots$$

(ii) *The first terms of the Laurent series for $(1+z)A_k(z)$ at $z=1$ are*

$$=(-1)^{\frac{d_2-d_1}{2}}\Big(-\frac{H_k-H_{d_1-k}+H_{k+\frac{d_2-d_1}{2}}-H_{\frac{d_1+d_2}{2}-k}}{2^{d_1+d_2}k!(d_1-k)!\left(\frac{d_2-d_1}{2}+k\right)!\left(\frac{d_2+d_1}{2}-k\right)!(1-z)^{d_1+d_2+1}}+$$
$$+\frac{d_1+d_2}{(1-z)^{d_1+d_2}\cdot 2^{d_1+d_2-1}k!(d_1-k)!\left(\frac{d_2-d_1}{2}+k\right)!\left(\frac{d_2+d_1}{2}-k\right)!}-$$
$$-\frac{(d_1+d_2+2)(d_1-2k+1)\left(H_k-H_{d_1-k}+H_{k+\frac{d_2-d_1}{2}}-H_{\frac{d_1+d_2}{2}-k}\right)}{(1-z)^{d_1+d_2}\cdot 2^{d_1+d_2+1}k!(d_1-k)!\left(\frac{d_2-d_1}{2}+k\right)!\left(\frac{d_2+d_1}{2}-k\right)!}\Big).$$

where $H=\sum_{i=1}^{n}\frac{1}{n}$ is the n-th harmonic number

Theorem 4. *Suppose d_1 and d_2 are odd and $d_2>d_1$; then the degree of the algebra of joint invariants of two binary forms $\deg(\mathcal{I}_{d_1,d_2})$ is equal to*

$$\deg\mathcal{I}_{d_1,d_2}(z)=h(d_1,d_2)+g(d_2,d_1,d_2-d_1-2)$$

where $h(b,d)=\frac{(-1)^{\frac{d-b}{2}}}{4b!d!}\sum_{k=0}^{b/2}(k-\frac{b}{2})^{b+d-2}\binom{b}{k}\binom{d}{\frac{d+b}{2}-k}\times$
$\times\left(H_k-H_{b-k}+H_{k+\frac{d-b}{2}}-H_{\frac{b+d}{2}-k}-\frac{b+d-2}{k-\frac{b}{2}}\right)$.

Доведення. Let us $L=H_k-H_{d_1-k}+H_{k+\frac{d_2-d_1}{2}}-H_{\frac{d_1+d_2}{2}-k}$. Using Lemmas 2 and 4, we have:

$$\sum_{\frac{d_1}{2}\leqslant k\leqslant d_1}\varphi_{2k-d_1}\left((1-z^2)A_k(z)\right)=$$
$$=(-1)^{\frac{d_2-d_1+1}{2}}\frac{\sum_{\frac{d_1}{2}\leqslant k\leqslant d_1}\binom{d_1}{k}\binom{d_2}{\frac{d_2+d_1}{2}-k}(k-\frac{d_1}{2})^{d_1+d_2-1}L}{2(1-z)^{d_1+d_2}d_1!d_2!}+$$
$$+(-1)^{\frac{d_2-d_1}{2}}\frac{\sum_{\frac{d_1}{2}\leqslant k\leqslant d_1}\binom{d_1}{k}\binom{d_2}{\frac{d_2+d_1}{2}-k}(2k-d_1)^{d_1+d_2-2}L}{2^{d_1+d_2}(1-z)^{d_1+d_2-1}d_1!d_2!}\times$$
$$\times((2k-d_1-1)(d_1+d_2)+(d_1+d_2+2)(d_1-2k+1))+$$
$$+(-1)^{\frac{d_2-d_1}{2}}\sum_{\frac{d_1}{2}\leqslant k\leqslant d_1}\frac{(2k-d_1)^{d_1+d_2-2}(d_1+d_2)}{(1-z)^{d_1+d_2-1}\cdot 2^{d_1+d_2-1}k!(d_1-k)!\left(\frac{d_2-d_1}{2}+k\right)!\left(\frac{d_2+d_1}{2}-k\right)!}+$$
$$+O((1-z)^{-d_1-d_2+1}).$$

In the same way, we obtain

$$\sum_{\frac{d_1}{2}\leqslant k\leqslant d_1}\left(z\varphi_{2k-d_1}\left(1-z^2\right)B_k(z)\right)\right)'_z=$$
$$=\frac{(-1)^{\frac{d_1-d_2}{2}}(d_1+d_2-1)}{2^{d_1+d_2-1}d_1!d_2!(1-z)^{d_1+d_2}}\sum_{\frac{d_1}{2}\leqslant k\leqslant d_1}(2k-d_1)^{d_1+d_2-2}\binom{d_1}{k}\binom{d_2}{\frac{d_2+d_1}{2}-k}-$$
$$-\frac{(-1)^{\frac{d_1-d_2}{2}}(d_1+d_2-2)}{2^{d_1+d_2}d_1!d_2!(1-z)^{d_1+d_2-1}}\sum_{\frac{d_1}{2}\leqslant k\leqslant d_1}\left((2k-d_1)^{d_1+d_2-2}+2(2k-d_1)^{d_1+d_2-3}\right)\binom{d_1}{k}\times$$
$$\times\binom{d_2}{\frac{d_2+d_1}{2}-k}+O((1-z)^{-d_1-d_2+2}).$$

Let us apply Theorem 3. Since $\operatorname{tr}\deg_{\mathbb{C}} \mathcal{I}_{d_1,d_2} = d_1 + d_2 - 1,$ it follows that coefficient of $\frac{1}{(1-z)^{d_2+d_1}}$ equals $0:$

$$f = -\frac{1}{d_1!d_2!} \sum_{\frac{d_1}{2} \leqslant k \leqslant d_1} (-1)^{\frac{d_2-d_1}{2}} \binom{d_1}{k} \binom{d_2}{\frac{d_2+d_1}{2} - k} (k - \frac{d_1}{2})^{d_1+d_2-1} L +$$

$$+ \frac{(-1)^{\frac{d_2-d_1}{2}}(d_1+d_2-1)}{2^{d_1+d_2-1}d_1!d_2!} \sum_{\frac{d_1}{2} \leqslant k \leqslant d_1} \left((2k-d_1)^{d_1+d_2-2} \binom{d_1}{k} \binom{d_2}{\frac{d_2+d_1}{2} - k} \right) +$$

$$+ \frac{1}{2^{d_2-1}d_2!} \sum_{k=0}^{\frac{d_2-d_1}{2}-1} \binom{d_2}{k} \frac{(-1)^{k+d_1+1}(d_2-2k)^{d_1+d_2-1}}{\prod_{i=0}^{d_1}(2k-d_1-d_2+2i)} +$$

$$+ \frac{1}{2^{d_2-1}d_2!} \sum_{k=\frac{d_1+d_2}{2}+1}^{d_2} \binom{d_2}{k} \frac{(-1)^{k+d_1+1}(d_2-2k)^{d_1+d_2-1}}{\prod_{i=0}^{d_1}(2k-d_1-d_2+2i)} = 0.$$

Adding the coefficients of $\frac{1}{(1-z)^{d_2+d_1-1}}$, we get

$$\deg \mathcal{I}_{d_1,d_2}(z) = \lim_{z \to 1} (1-z)^{d_2+d_1-1} \mathcal{P}(\mathcal{I}_{d_1,d_2}, z) =$$

$$= f + \frac{1}{4d_1!d_2!} \sum_{\frac{d_1}{2} \leqslant k \leqslant d_1} (-1)^{\frac{d_2-d_1}{2}} \binom{d_1}{k} \binom{d_2}{\frac{d_2+d_1}{2} - k} (k - \frac{d_1}{2})^{d_1+d_2-2} L -$$

$$- \frac{(-1)^{\frac{d_2-d_1}{2}}(d_1+d_2-2)}{2^{d_1+d_2-1}d_1!d_2!} \sum_{\frac{d_1}{2} \leqslant k \leqslant d_1} \left((2k-d_1)^{d_1+d_2-3} \binom{d_1}{k} \binom{d_2}{\frac{d_2+d_1}{2} - k} \right) -$$

$$- \frac{1}{2^{d_2-1}d_2!} \sum_{0 \leqslant 2k \leqslant d_2-d_1-2} \binom{d_2}{k} \frac{(-1)^{k+d_1+1}(d_2-2k)^{d_1+d_2-2}}{\prod_{i=0}^{d_1}(2k-d_1-d_2+2i)}.$$

To conclude the proof, it remains to note that $f = 0.$ □

By Lemma 3, poles of functions $(1-z^2)A_k(z)$, $(1-z^2)B_k(z)$ and $(1-z^2)C_k(z)$ at point $z=-1$ give poles of functions $\varphi_{2k-d_1}\left((1-z^2)A_k(z)\right)$, $\varphi_{2k-d_1}\left((1-z^2)B_k(z)\right)$ and $\varphi_{d_2-2k}\left((1-z^2)C_k(z)\right)$ at $z=1$ respectively. It now follows the following theorem

Theorem 5. *Suppose d_1 and d_2 are even and $d_2 > d_1$; then the degree of the algebra of joint invariants of two binary forms $\deg(\mathcal{I}_{d_1,d_2})$ is equal to*

$$\deg \mathcal{I}_{d_1,d_2}(z) = 2\left(h(d_1,d_2) + g(d_2,d_1,d_2-d_1-2)\right).$$

References

[1] Bedratyuk, L. The Poincaré series for the algebra of covariants of a binary form // Int. J. Algebra. — 2010. — V.4 — №25-28— P. 1201–1207.

[2] Bedratyuk L. The Poincaré series of the algebras of joint invariants and covariants of two binary forms // Linear and Multilinear Algebra. — 2010. — V.58 — №6. P. 789–803.

[3] Bedratyuk L., Ilash N. The degree of the algebra of covariants of a binary form // J. Commut. Algebra — 2015. — V.7 — №4 — P. 459–472.

[4] *Benson D.* Polynomial invariants of finite groups. London Mathematical Society Lecture Note Series. 190. — Cambridge University Press, 1993. — 118 p.

[5] Hilbert D. Über die vollen Invariantsystemes // Mathem. Annalen — 1893 — №42 — P. 313–373.

[6] Luna D. Sur les orbites fermées des groupes algébriques róductifs // Invent. Math. — 1972 — №16 — P. 1–5.

[7] *Popov V. L.* Groups, Generators, Syzygies, and Orbits in Invariant Theory // Translations of Mathematical Monographs. 100. — American Mathematical Society, Providence, RI, 1992 —- P. 245.

[8] *Springer T.A.* Invariant theory. Lecture Notes in Mathematics. 585. — Berlin: Springer-Verlag, 1977. — 111 p.

[9] Springer T.A. On the invariant theory of SU_2 // Indag. Math. — 1980 — №42 — P. 339-345.

[10] Ілаш Н.Б. Степiнь алгебри коварiантiв двох бiнарних форм // Вiсник Донецького нацiонального унiверситету. Сер. А: Природничi науки. — 2015 —№1-2— С. 37–45.

СТЕПІНЬ АЛГЕБРИ СПІЛЬНИХ ІНВАРІАНТІВ ДВОХ БІНАРНИХ ФОРМ

Ілаш Н.Б.

РЕЗЮМЕ

Обчислюється степінь алгебри спільних інваріантів двох бінарних форм. Для цього використано ряд Пуанкаре цієї алгебри.

Ключові слова: інваріант, бінарна форма, ряд Пуанкаре, алгебра спільних інваріантів, степінь алгебри.

СТЕПЕНЬ АЛГЕБРЫ ОБЩИХ ИНВАРИАНТОВ ДВУХ БИНАРНЫХ ФОРМ

Илаш Н.Б.

РЕЗЮМЕ

Исчисляется степень алгебры общих инвариантов двух бинарных форм. Для этого использовано ряд Пуанкаре этой алгебры.

Ключевые слова: инвариант, бинарная форма, ряд Пуанкаре, алгебра общих инвариантов, степень алгебры.