УДК 512.562

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A CLASSIFICATION OF GENERALIZED FUNCTIONAL EQUATIONS ON TERNARY QUASIGROUPS

A classification of generalized ternary functional equations on invertible functions defined on an arbitrary set of the functional lengths one and two up to parastrophically primary equivalence has been found.

Key words: *ternary quasigroup, invertible function, functional equation, parastrophically primary equivalence.*

Inroduction

We continue classification of functional equations on invertible functions, i.e., quasigroup operations [2, 5, 4, 7, 8, 9, 10, 11, 12].

A functional equation is understood as a universal quantified equality of two second order terms consisting of functional and individual variables. The number of all functions is called a *functional length*. A functional equation is called: *generalized*, if all functional variables are pairwise different; *ternary*, if all functional variables are ternary; *quasigroup*, if all functional variables are supposed to take their values in sets of invertible operations. A ternary operation f defined on an arbitrary set Q is called invertible, if for all elements a and b each of the terms $f(x,a,b)$, $f(a,x,b)$, $f(a,b,x)$ establishes a permutation of Q.

Some functional equations are equivalent to a systems of shorter equations. That is why it is very important to classify functional equations of small lengths. In this article, we give full classification of generalized ternary functional equations of the length one (Theorem 1.) and of the length two (Theorem 2.). The classification is done up to parastrophically primary equivalence (two functional equations are parastrophically primarily equivalent if one can be obtained from the other in a finite number of the following steps: renaming functional and individual variables, applying the identities which define the invertibility of operations). Moreover, we have found full solution of each functional equation from a transversal of classes of this equivalence.

1. Preliminaries

An n-ary operation f defined on a set Q is an assignment of exactly one element b from Q to every n-tuple $(a_0,\dots,a_{n-1})$ of elements from Q. In other words, f is a mapping from Q^n to Q. The set Q is called a *carrier*, n is an *arity* of f and we write $f(a_0,\dots,a_{n-1})=b$. The set of all n-ary operations defined on Q is denoted by O_n. Multiplications of n-ary operations is defined by

$$(f \underset{i}{\oplus} g)(x_0,\dots,x_{n-1}) := f(x_0,\dots,x_{i-1},g(x_0,\dots,x_{n-1}),x_{i+1},\dots,x_{n-1}) \qquad (1)$$
$$i=0,1,\dots,n-1.$$

It is easy to verify, that every of above defined superpositions is associative. Therefore, the algebra $(O_n; \underset{i}{\oplus}, e_i)$ is a monoid, where $e_i(x_0,\dots,x_{n-1}) := x_i$ is its neutral element. The monoid $(O_n; \underset{i}{\oplus}, e_i)$ will be called i-th *symmetric monoid* of n-ary operations.

An n-ary function f is called:

- i*-th invertible*, if it is invertible in the i-th symmetric monoid, the inverse ${}^{[i]}f$ will be also called an i*-th division* of f;
- *invertible*, if it is i-th invertible for all $i = 0,\dots,n-1$;
- *left invertible*, if it is 0-th invertible;
- *right invertible*, if it is $(n-1)$-th invertible.

If an operation f is invertible, then the algebra $(Q; f, {}^{[0]}f, \dots, {}^{[n-1]}f)$ is called a *quasigroup* or an n*-ary quasigroup*. Consequently, an algebra $(Q; f, {}^{[0]}f, \dots, {}^{[n-1]}f)$ is called a quasigroup, if it satisfies the following identities:

$$\begin{aligned} f(x_0,\dots,x_{i-1}, {}^{[i]}f(x_0,\dots,x_{n-1}), x_{i+1},\dots,x_{n-1}) &= x_i, \\ {}^{[i]}f(x_0,\dots,x_{i-1}, f(x_0,\dots,x_{n-1}), x_{i+1},\dots,x_{n-1}) &= x_i, \\ i &= 0,\dots,n-1. \end{aligned} \tag{2}$$

Denote $x_n := f(x_0,\dots,x_{n-1})$, then (2) is equivalent to

$$ {}^{[i]}f(x_0,\dots,x_{i-1},x_n,x_{i+1},\dots,x_{n-1}) = x_i \quad \Leftrightarrow \quad f(x_0,\dots,x_{n-1}) = x_n. $$

Giving a character $[i]$ the value (i,n), i.e., a permutation of the set $\{0,1,\dots,n\}$, we obtain

$$ {}^{(i,n)}f(x_0,\dots,x_{i-1},x_{i(i,n)},x_{i+1},\dots,x_{n-1}) = x_{n(i,n)} \quad \Leftrightarrow $$

$$ \Leftrightarrow \quad f(x_0,\dots,x_{n-1}) = x_n. $$

The cycles $(0,n)$, ..., $(n-1,n)$ generate the symmetric group S_{n+1} of the set $\{0,1,\dots,n\}$. Therefore, there exist $(n+1)!$ operations to every n-ary invertible operation and all of them are defined by the relationship

$$ {}^{\sigma}f(x_{0\sigma},\dots,x_{(n-1)\sigma}) = x_{n\sigma} \quad \Leftrightarrow \quad f(x_0,\dots,x_{n-1}) = x_n, $$

where σ belong to the symmetry group S_{n+1} of the set $\{0,1,\dots,n\}$. All of the operations are called *parastrophes* of f and ${}^{\sigma}f$ is called σ*-parastrophe* of f. It is easy to verify the validity of the formula

$$ {}^{\sigma}\left({}^{\tau}f\right) = {}^{\sigma\tau}f $$

for every invertible operation f and all permutations $\sigma,\tau \in S_{n+1}$. This relationship means that the group S_{n+1} acts on the set of all n-ary invertible operations Δ_n. In particular, the parastrophic symmetry group $\mathrm{Ps}(f)$ defined by

$$ \mathrm{Ps}(f) := \{\sigma \mid {}^{\sigma}f = f\} $$

is a stabilizer under this action. Thus, parastrophes ${}^{\tau}f$ and ${}^{\sigma}f$ coincide if and only if $\tau \in \sigma\mathrm{Ps}(f)$, so, the set of all different parastrophes of the given invertible operation f is

$$ \{{}^{\sigma}f \mid \sigma \in \mathrm{T}\}, $$

where T is a coset transversal for the sub-group $\mathrm{Ps}(f)$ of the group S_{n+1}. Consequently, the number of different parastrophes of an invertible operation f equals the index of its parastrophic symmetry group in S_{n+1}, i.e., $(n+1)!/|\mathrm{Ps}(f)|$.

Unary quasigroups ($n=1$). There is only one symmetric monoid $(O_1;\underset{0}{\oplus},e_0)$. It is a well-known symmetric monoid of transformations of the set Q: $(\mathfrak{S};\circ,\iota)$. Therefore, every unary invertible operation f has exactly one inverse ${}^{[0]}f=f^{-1}$ and the defining equality (2) means

$$f\circ f^{-1}=f^{-1}\circ f=\iota.$$

Thus, an algebra $(Q;f,f^{-1})$ is a unary quasigroup, if the identities

$$f(f^{-1}(x))=x,\qquad f^{-1}(f(x))=x$$

hold. An arbitrary unary inverse operation f has $2!=2$ parastrophes, f and f^{-1}.

Binary quasigroups ($n=2$). There are two symmetric monoids: $(O_2;\underset{0}{\oplus},e_0)$ and $(O_2;\underset{1}{\oplus},e_1)$, where O_2 is the set of all binary operations defined on Q; the superpositions $\underset{0}{\oplus}$ and $\underset{1}{\oplus}$ are left and right multiplications of binary operations:

$$(f\underset{0}{\oplus}g)(x,y)=f(g(x,y),y),\qquad (f\underset{1}{\oplus}g)(x,y)=f(x,g(x,y));$$

the operations e_0 and e_1 are selectors: $e_0(x,y):=x$, $e_1(x,y):=y$. Therefore, an algebra $(Q;f,{}^{\ell}f,{}^{r}f)$ is a quasigroup if the following identities

$$f({}^{\ell}f(x,y),y)=x,\quad {}^{\ell}f(f(x,y),y)=x,$$
$$f(x,{}^{r}f(x,y))=y,\quad {}^{r}f(x,f(x,y))=y$$

are true, where $\ell:=[0]:=(13)$, $r:=[1]:=(23)$, $s:=(12)$. Consequently, $S_3:=\{\iota,\ell,r,s,s\ell,sr\}$.

Ternary quasigroups ($n=3$). There are three symmetric monoids:

$$(O_3;\underset{0}{\oplus},e_0),\quad (O_3;\underset{1}{\oplus},e_1),\quad (O_3;\underset{2}{\oplus},e_2),$$

where O_3 is the set of all ternary operations defined on Q; the superpositions $\underset{0}{\oplus}$, $\underset{1}{\oplus}$ and $\underset{2}{\oplus}$ are left, middle and right multiplications of ternary operations:

$$(f\underset{0}{\oplus}g)(x,y,z)=f(g(x,y,z),y,z),\quad (f\underset{1}{\oplus}g)(x,y,z)=f(x,g(x,y,z),z),$$
$$(f\underset{2}{\oplus}g)=f(x,y,g(x,y,z));$$

the operations e_0, e_1, e_2 are selectors:

$$e_0(x,y,z):=x,\quad e_1(x,y,z):=y,\quad e_2(x,y,z):=z.$$

Therefore, an algebra $(Q;f,{}^{[0]}f,{}^{[1]}f,{}^{[2]}f)$ is a quasigroup, if the following identities

$$\left.\begin{array}{ll} f({}^{[0]}f(x,y,z),y,z)=x, & {}^{[0]}f(f(x,y,z),y,z)=x,\\ f(x,{}^{[1]}f(x,y,z),z)=y, & {}^{[1]}f(x,f(x,y,z),z)=y,\\ f(x,y,{}^{[2]}f(x,y,z))=z, & {}^{[2]}f(x,y,f(x,y,z))=z \end{array}\right\}\qquad (3)$$

are true.

Some definitions. Let δ_f denote the *main diagonal* the cube of its values, i.e., $\delta_f(x) := f(x,\dots,x)$. An operation f is called *main diagonal*, if δ_f is a permutation of the carrier.

For an arbitrary ternary operation f, we define binary operations $(\underset{l}{\circ})$, $(\underset{m}{\circ})$, $(\underset{r}{\circ})$ by the equalities

$$x \underset{l}{\circ} y := f(x,y,y), \qquad x \underset{m}{\circ} y := f(y,x,y), \qquad x \underset{r}{\circ} y := f(y,y,x) \tag{4}$$

and we will called them *left*, *middle* and *right diagonal operations* respectively.

An ternary operation f will be called *left*, *middle* and *right neutral operation*, if it satisfies the respective identity:

$$f(x,y,y) = x, \qquad f(y,x,y) = x, \qquad f(y,y,x) = x. \tag{5}$$

A ternary quasigroup is said to be *totally symmetric*, if all parastrophes coincide. Following [6], a neutral totally symmetric quasigroup will be called a *Steiner quasigroup*.

Functional equations. In this articles, we study functional equations which can be considered on arbitrary sets. Namely, let T_1 and T_2 be terms which consist of functional and individual variables, i.e. they do not have functional and individual constants. Universally quantified formula $T_1 = T_2$ is called a *functional equation* see[1, 3].

A value of lexicographic sequence of all free functional variables of the given functional equation is called its *solution*, if the equation becomes an identity after substituting the value for functional variables. The set of all solutions defined on the same set is called a *solution set* of the equation.

A functional equation is called:

1. *quasigroup* if its functional variables present quasigroup operations;

2. *generalized*, if all its functional variables are pairwise different;

3. *trivial*, if it has solutions only on one-element sets.

For example, a functional quasigroup equation is trivial, if one of its individual variables has only one appearance.

Two functional equations are called:

1. *equivalent*, if they have the same solution set on every carrier;

2. *parastrophically primarily equivalent*, if one can be obtained from the other in a finite number of the following steps:

   (a) applying the quasigroup hyperidentities (3);
   (б) changing the sides of the equation;
   (в) renaming the individual variables;
   (г) renaming the functional variables.

Each of these items does not change the number of different individual variables. Therefore, if functional equations are parastrophically primarily equivalent, then they have the same number of different individual variables. In other words, the following assertion is true.

Lemma 1. *If functional equations do not have the same number of different individual variables, then they are not parastrophically primarily equivalent.*

Lemma 2. *Let functional equations* $\omega_1 = \upsilon_1$ *and* $\omega_2 = \upsilon_2$ *be parastrophically primarily equivalent. Then there exist permutations* $\sigma_0, \ldots, \sigma_{m-1}$ *of the set* $\{0,1,2,3\}$ *and a permutation* τ *of* $\{0,\ldots,m-1\}$ *such that for every solution* $(f_0,\ldots,f_{m-1})$ *the tuple* $({}^{\sigma_0}f_{0\tau},\ldots,{}^{\sigma_{m-1}}f_{(m-1)\tau})$ *is a solution of* $\omega_2 = \upsilon_2$.

Let a functional equation contain m independent variables. We say that a quasigroup $(Q;f)$ is a solution of $\omega = \upsilon$ if the tuple $(\underbrace{f,\ldots,f}_{m \text{ times}})$ is its solution.

Corollary 1. *If a totally symmetric quasigroup is a solution of a functional equation and it is not a solution of another equation, then the functional equations are not parastrophically primarily equivalent.*

2. Classification of ternary functional equations

Functional equations of the length 1. Since every non-trivial quasigroup functional equation has at least two appearances of each individual variable, the following statement is true.

Theorem 1. *Every ternary quasigroup functional equation is parastrophically primarily equivalent to an exactly one of the equations:*

$$F(x,x,x) = x, \quad (i) \qquad\qquad F(x,y,y) = x. \quad (ii) \tag{6}$$

Proof. A quasigroup functional equation in one a functional variable has the form

$$F(u_1,u_2,u_3) = u_4,$$

where the set $\{u_1,u_2,u_3,u_4\}$ consists of one or two different individual variables. If it has one variable, then the equation is (i). Otherwise, renaming the individual variables, we obtain one of the following functional equations:

$$F(x,y,y) = x, \quad (a) \quad F(y,x,y) = x, \quad (b) \quad F(y,y,x) = x. \quad (c) \tag{7}$$

Permute the first and the second variables in (b) and also the first and the third variables in (c):

$${}^{(12)}F(x,y,y) = x, \quad (b') \qquad {}^{(13)}F(x,y,y) = x. \quad (c')$$

Replacing ${}^{(12)}F$ and ${}^{(13)}F$ with F, we obtain (ii).

Suppose, the equations (i) and (ii) are parastrophically primarily equivalent. Therefore, for every solution f of (i) some parastrophe of f is a solution of (ii). Let $\mathbb{Z}_5$ be the ring of integers modulo 5. An operation f defined by

$$f(x,y,z) := 2x + 2y + 2z$$

is a solution of the quasigroup equation (i):

$$f(x,x,x) = 2x + 2x + 2x = 6x = x.$$

The operation f is invertible and all its divisions are

$$^{(12)}f(x,y,z) = 3x - y - z, \qquad ^{(13)}f(x,y,z) = -x + 3y - z,$$
$$^{(14)}f(x,y,z) = -x - y + 3z.$$

For example,

$$f(^{(12)}f(x,y,z),y,z) = 2(3x - y - z) + 2y + 2z = 6x - 2y - 2z + 2y + 2z = x.$$

All other equalities are proved in the same way. It is easy to see that $S_3 \subseteq \mathrm{Ps}(f)$, then f has not more than four different parastrophes. The operations f, $^{(14)}f$, $^{(24)}f$, $^{(34)}f$ are evidently pairwise different and they are all parastrophes of f. But none of them is a solution of (ii):

$$f(0,1,1) = 0 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 = 4 \neq 0, \quad ^{(13)}f(0,1,1) = -0 + 3 - 1 = 2 \neq 0,$$
$$^{(12)}f(0,1,1) = 0 - 1 - 1 = 3 \neq 2, \qquad ^{(14)}f(0,1,1) = -0 - 1 + 3 = 2 \neq 0.$$

A contradiction (see Lemma 2.). Thus, the functional equations (i) and (ii) are not parastrophically primarily equivalent. □

Functional equations of the length 2. For these functional equations the following theorem is true.

Theorem 2. *Every ternary quasigroup functional equation is parastrophically primarily equivalent to exactly one of the equations:*

$$F_1(x,x,x) = F_2(x,x,x), \tag{8}$$
$$F_1(x,x,x) = F_2(x,y,y), \tag{9}$$
$$F_1(x,x,y) = F_2(x,x,y), \tag{10}$$
$$F_1(x,x,x) = F_2(y,y,y), \tag{11}$$
$$F_1(x,x,y) = F_2(x,y,y), \tag{12}$$
$$F_1(x,x,y) = F_2(y,z,z), \tag{13}$$
$$F_1(x,y,z) = F_2(x,y,z). \tag{14}$$

Proof. Every ternary functional equation in two functional variables is a formula:

$$F_1(u_1,u_2,u_3) = F_2(u_4,u_5,u_6), \tag{a}$$

if functional variables are in both sides of the equality and one of the formulas

$$F_1(F_2(u_1,u_2,u_3),u_4,u_5) = u_6, \qquad F_1(u_1,F_2(u_2,u_3,u_4),u_5) = u_6,$$

$$F_1(u_1,u_2,F_2(u_3,u_4,u_5)) = u_6,$$

if functional variables are in the same side. The last three equalities are reduced to (a) using (3).

Thence, it is enough to consider the case (a). Since a non-trivial quasigroup functional equation is under consideration (a), then every individual variable has at least two appearances. Therefore, the set $\{u_1,\ldots,u_6\}$ say, a *variable set*, can have one, two or three elements.

If the variable set has one element, then the equation is (8). If the set has three elements, then every individual variable appears twice. Consequently, there are two possibilities: the left side of the equality has two different variables and then it is (13), and it has three different variables and then it is (14).

Let the variable set have two elements. Therefore, there are two possibilities:

- one of the variables, say y, appears twice. If the appearances are in the same side, then it is the identity (9) otherwise it is (10);

- each of the individual variables appease thrice. If all appearances of a variable are in the same side, then the equation is (11). Otherwise, it is (12).

It remains to prove the non-equivalency of the equations (8)–(12). The proof is given in the following table.

	(9)	(10)	(11)	(12)	(13)	(14)
(8)	Lem 1.	Lem 1.	Lem 1.	Lem 1.	Lem 1.	Lem 1.
(9)	×	Ex 1	Steiner	Steiner	Lem 1.	Lem 1.
(10)	×	×	Steiner	Steiner	Lem 1.	Lem 1.
(11)	×	×	×	Ex 2	Lem 1.	Lem 1.
(12)	×	×	×	×	Lem 1.	Lem 1.
(13)	×	×	×	×	×	Ex 3

If in the cell which is on the intersection of (i)-th row and (j)-th column is 'Steiner', then an arbitrary nontrivial ternary Steiner quasigroup is a solution of one of the equations (i), (j) and it is not a solution of the other equation. According to Corollary 1., they are not parastrophically primarily equivalent. For example, let $i=10$, $j=11$ and let $(Q;f)$ be a ternary Steiner quasigroup. Since a ternary Steiner quasigroup satisfies the identity $f(x,x,y)=y$, then the pair (f,f) is a solution of the equation (10). According to Lemma 2., there are two permutations σ, τ of the set $\{0,1,2,3\}$ such that the pair $({}^{\sigma}f,{}^{\tau}f)$ should be a solution of the equation (11). By definition, a Steiner quasigroup is totally symmetric, i.e., all of its parastrophes coincide. Therefore, the pair (f,f) should be a solution of (11), that is $f(x,x,x)=f(y,y,y)$ is an identity. Since a Steiner quasigroup is idempotent, this identity is equivalent to $x=y$. Consequently, the quasiqroup $(Q;f)$ is trivial. The obtained contradiction implies that the pair (f,f) is not a solution of (11). Thus, by the Corollary 1. the functional equations (10) and (11) are not parastrophicaly primarily equivalent.

If in the cell which is on the intersection of (i)-th row and (j)-th column is 'Lemma 1.', then the equation (i) and (j) have different number of different individual variables and according to Lemma 1., they are not parastrophically primarily equivalent.

(9), (10), Example 1. Let $\mathbb{Z}_3$ be the ring modulo 3 and let $h(x,y,z):=x+y+z$.

Suppose the equations (9), (10) are parastrophically primarily equivalent. According to Lemma 2., there exist permutations σ, π of $\{0,1,2,3\}$ such that for each solution (f,f) of the equation (10) the pair $({}^{\sigma}f,{}^{\pi}f)$ is a solution of (9).

Since the group $(\mathbb{Z}_3;+)$ is commutative, then the parastrophic symmetry group of the operation h includes S_3. Consequently, the operation h has not more than 4 different parastrophes. It is easy to verify, that

$$^{(03)}h(x,y,z)=-x+y+z,\quad ^{(13)}h(x,y,z)=x-y+z,\quad ^{(23)}h(x,y,z)=x+y-z.$$

Therefore, all different parastrophes of h are h, ${}^{(03)}h$, ${}^{(13)}h$, ${}^{(23)}h$.

It is easy to see, that the pair $({}^{\sigma^{-1}}h,{}^{\sigma^{-1}}h)$ is a solution of (10), then the pair

$$({}^{\sigma\sigma^{-1}}h,{}^{\pi\sigma^{-1}}h)=(h,{}^{\pi\sigma^{-1}}h)$$

is a solution of (9). It means that the identity

$$0 = {}^{\pi\sigma^{-1}}h(x, y, y)$$

holds. It is enough to consider the last identity for $\pi\sigma^{-1} \in \{\iota, (01), (02), (03)\}$. In these cases, we have

$$0 = x + 2y, \quad 0 = -x + 2y, \quad 0 = x, \quad 0 = x.$$

But it is possible only if the carrier is singleton, i.e., the solution is trivial. The obtained contradiction proved that the functional equations (9) and (10) are not parastrophically primarily equivalent.

(11), (12), Example 2. Let an operation h be defined as in Example 1.

Suppose, the equations (11), (12) are parastrophically primarily equivalent, then there exist permutations σ, τ such that for every solution (f, f) of the equation (11) the pair $({}^\sigma f, {}^\tau f)$ is a solution of (12). Let us prove that these is wrong.

It is easy to see that (h, h) is a solution of (11), then the pair $({}^\sigma h, {}^\tau h)$ is a solution of (9), i.e., for all $x, y \in \mathbb{Z}_3$ we have

$$ {}^\sigma h(x, x, y) = {}^\tau h(x, y, y). \tag{15}$$

Taking into account that h, ${}^{(03)}h$, ${}^{(13)}h$, ${}^{(23)}h$ are all different parastrophes of h, we obtan

$$ {}^\sigma h(x, x, y) = \begin{cases} x + 2y, & \text{if } \sigma = \iota, \\ 2y, & \text{if } \sigma = (14) \text{ or } \sigma = (24), \\ x + y, & \text{if } \sigma = (34), \end{cases}$$

$$ {}^\tau h(x, x, y) = \begin{cases} 2x + y, & \text{if } \tau = \iota, \\ x + y, & \text{if } \tau = (14), \\ 2x, & \text{if } \tau = (24) \text{ or } \sigma = (34). \end{cases}$$

In all these cases, (15) implies a contradiction. Indeed,

(13), (14), Example 3. Let $(Q; +)$ be an arbitrary group of exponent two, $(Q; \cdot)$ be a group, which is not isomorphic to $(Q; +)$. Define operations h and g:

$$h(x, y, z) := x + y + z, \qquad g(x, y, z) := x \cdot y \cdot z^{-1}, \tag{16}$$

then the quasigroup $(Q; h)$ is Steiner. The pair (g, h) is a solution of (13). If the pair $({}^\sigma g, {}^\tau h)$ or the pair $({}^\tau h, {}^\sigma g)$ for some σ, τ is a solution of (14), then the quasigroups $(Q; h)$ and $(Q; g)$ are parastrophic. Therefrom, the groups $(Q; +)$ and $(Q; \cdot)$ are isomorphic. A contradiction. Thus, the equations (13) and (14) are not parastrophically primarily equivalent. □

3. Quasigroup solutions of (8)-(14)

Proposition 1. *A pair (f, g) of invertible functions is a solution of the equation (8) if and only if their main diagonals are the same.*

Proof. A pair (f, g) is a solution of (8) on a carrier Q if and only if the identity

$$f(x, x, x) = g(x, x, x)$$

holds for all $x \in Q$. This relationship is equivalent to $\delta_f = \delta_g$. □

Proposition 2. *A pair of invertible functions* (f_1, f_2) *defined on* Q *is a solution of (9) if and only if there exists a left neutral invertible operation* h *and a permutation* δ *of* Q *such that*

$$f_1(x,x,x) = \delta(x), \qquad f_2(x,y,z) = \delta\, {}^{(14)}h(x,y,z).$$

Proof. Let a pair (f_1, f_2) of invertible operations is a solution of the equation (9), i.e., the identity

$$f_1(x,x,x) = f_2(x,y,y)$$

holds. Therefrom, $f_2(x,y,y) = \delta_1(x)$ therefore

$$^{(14)}f_2(\delta_1(x), y, y) = x. \tag{17}$$

Define an operation h

$$h(x,y,z) := \ ^{(14)}f_2(\delta_1(x), y, z). \tag{18}$$

The relationship (17) implies that the operation h is left neutral. But from (18) it follows that it is isostrophic to f_2, that is why it is invertible.

Vice versa, note if h is left neutral, then $^{(14)}h$ is left neutral as well. Therefore,

$$f_2(x,y,y) = \delta x, \qquad ^{(14)}h(x,y,y) = \delta(x) = f_1(x,x,x).$$

□

Proposition 3. *A pair of invertible operations is a solution of the functional equation (10) if and only if their left diagonals coincide.*

A ternary operation f defined on Q is called *unipotent*, if there exists an element $a \in Q$ such that for all $x \in Q$ the equality $f(x,x,x) = a$ holds.

Proposition 4. *A pair of ternary invertible operations is a solution of the functional equation (11) if and only if they are unipotent and the element of unipotency is common.*

Proposition 5. *A pair of ternary invertible operations* (f, g) *is a solution of the functional equation (12) if and only if a dual operation to the left diagonal of the operation* f *coincides with the left diagonal of the operation* g*.*

Proof. Let a pair of ternary invertible operations (f_1, f_2) is a solution of the functional equation (13), i.e.,

$$f_1(x,x,y) = f_2(y,z,z).$$

In particular when $z = a \in Q$, we have

$$f_1(x,x,y) = \alpha y$$

for some permutation α of the set Q. This equality implies

$$f_1(x,x,\alpha^{-1}y) = y.$$

Hence, an operation h defined by

$$h(x,y,z) := f_1(x,y,\alpha^{-1}z)$$

is right neutral. It is invertible because it is isotopic to an invertible operation f_1. Therefrom

$$f_1(x,y,z) = h(x,y,\alpha z).$$

Thus, $f_2(y,z,z) = \alpha y = \alpha h(x,x,y)$. □

Proposition 6. *A pair of ternary invertible operations* (f_1, f_2) *is a solution of the functional equation (13) if and only if there exists a right neutral operation* h *and a permutation* α *such that*

$$f_1(x, y, z) = h(x, y, \alpha z), \qquad f_2(y, z, z) = \alpha h(x, x, y).$$

Proposition 7. *A pair of operations is a solution of the functional equation (14) if and only if they coincide.*

Conclusion

Classifying functional ternary functional equations on invertible functions defined on an arbitrary set, the following results have been obtained: there exist two ternary functional equations of the length one (Theorem 1.) and seven functional equations of the length two (Theorem 2.).

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КЛАСИФІКАЦІЯ УЗАГАЛЬНЕНИХ ФУНКЦІЙНИХ РІВНЯНЬ НА ТЕРНАРНИХ КВАЗІГРУПАХ

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РЕЗЮМЕ

З точністю до парастрофно первинної еквівалентності встановлено класифікацію узагальнених тернарних функційних рівнянь функційної довжини 1 і 2 на оборотних функціях, заданих на довільному носієві.

Ключові слова: *тернарна квазігрупа, оборотна функція, функційне рівняння, парастрофно первинна рівносильність*

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КЛАССИФИКАЦИЯ ОБОБЩЕННЫХ ФУНКЦИОНАЛЬНЫХ УРАВНЕНИЙ НА ТЕРНАРНЫХ КВАЗИГРУППАХ

РЕЗЮМЕ

С точностью к первично парастрофной эквивалентности, найдена классификация обобщенных тернарных функциональных уранений функциональной длины 1 и 2 на обратимых функциях, которые заданы на произвольном множестве.

Ключевые слова: *тернарная квазигруппа, обратимая функция, функциональное уравнение, первично парастрофная эквивалентность.*