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ON LINEARITY OF ISOTOPES OF ABELIAN GROUPS

The concept of middle linearity for isotopes of Abelian groups is introduced. If a quasigroup is one-sided linear then two of its parastrophes are left linear, two right linear and two are middle linear. The corresponding formula is given. Relations for the corresponding varieties and identities are established. Involutional one-sided central quasigroups are characterized.

Keywords: *quasigroup, linear quasigroup, group isotope.*

Introduction

This article is a continuation of [4, 5, 1]. Throughout the article only isotopes of commutative groups are under consideration, i.e., the word '*linear*' could be replaced by '*central*'.

Remind that a quasigroup $(Q;\circ)$ defined by $x\circ y := \alpha x+\beta y+a$ and $\alpha 0=\beta 0=0$ for some group $(Q;+,0)$, is called: 1) *left linear* if α is an automorphism of $(Q;+)$; 2) *right linear* if β is an automorphism of $(Q;+)$; 3) *linear* if both α and β are automorphisms of $(Q;+)$. Let $(Q;\circ)$ be strictly left linear quasigroup, that is, α is an automorphism of $(Q;+)$ but β is not. It is easy to prove that its (23)-parastrophe and (132)-parastrophe are neither left linear, nor right linear. The property which corresponds to the left linearity is called here *middle linearity.* Namely, $(Q;\circ)$ is called *middle linear* if $\alpha\beta^{-1}$ is an automorphism of the commutative group $(Q;+)$.

In Theorem 4. and Corollary 2. a dependence among left, right and middle linearities is established.

1. Preliminaries

A *quasigroup* is an algebra $(Q;\cdot,\overset{\ell}{\cdot},\overset{r}{\cdot})$ satisfying the identities

$$(x\cdot y)\overset{\ell}{\cdot} y=x,\quad (x\overset{\ell}{\cdot} y)\cdot y=x,\quad x\overset{r}{\cdot}(x\cdot y)=y,\quad x\cdot(x\overset{r}{\cdot} y)=y. \tag{1}$$

The operation $(\cdot)$ is called *main* and $(\overset{\ell}{\cdot})$, $(\overset{r}{\cdot})$ are its *left* and *right divisions.* These operations and their dual, which are defined by

$$x\overset{s}{\cdot} y:=y\cdot x,\qquad x\overset{s\ell}{\cdot} y:=y\overset{\ell}{\cdot} x,\qquad x\overset{sr}{\cdot} y:=y\overset{r}{\cdot} x \tag{2}$$

are called *parastrophes* of $(\cdot)$ and defining identities are called *primary.* Relationships (2) establish bijection among identities of signatures $(\cdot,\overset{\ell}{\cdot},\overset{r}{\cdot})$ and identities of $(\cdot,\overset{\ell}{\cdot},\overset{r}{\cdot},\overset{s}{\cdot},\overset{s\ell}{\cdot},\overset{sr}{\cdot})$. Therefore, throughout of the article we consider identities on quasigroups of signature $(\cdot,\overset{\ell}{\cdot},\overset{r}{\cdot},\overset{s}{\cdot},\overset{s\ell}{\cdot},\overset{sr}{\cdot})$. All parastrophes of $(\cdot)$ can be also defined by:

$$x_{1\sigma}\overset{\sigma}{\cdot} x_{2\sigma}=x_{3\sigma} :\Leftrightarrow\ x_1\cdot x_2=x_3,$$

where $\sigma \in S_3 := \{\iota, \ell, r, s, s\ell, sr\}$, $s := (12)$, $\ell := (13)$, $r := (23)$. It is easy to verify that

$$^{\sigma}\!\left(\overset{\tau}{\cdot}\right) = \left(\overset{\sigma\tau}{\cdot}\right)$$

holds for all $\sigma, \tau \in S_3$.

Transformation of an identity $\mathfrak{id}$ to identity $\mathfrak{id}'$ using primary identities is called a *primary (parastrophic) transformation* [2].

Let P be an arbitrary proposition in a class of quasigroups $\mathfrak{A}$. The proposition ${}^{\sigma}P$ is said to be a σ-parastrophe of P, if it can be obtained from P by replacing in every τ-parastrophe the symbol τ with $\tau\sigma^{-1}$ for $\tau \in S_3$; ${}^{\sigma}\mathfrak{A}$ denotes the class of all σ-parastrophes of quasigroups from $\mathfrak{A}$.

Theorem 1. [3] *Let $\mathfrak{A}$ be a class of quasigroups, then a proposition P is true in $\mathfrak{A}$ if and only if ${}^{\sigma}P$ is true in ${}^{\sigma}\mathfrak{A}$.*

Definition 1. *Transformation of the identity $\mathfrak{id}$ to the identity ${}^{\sigma}\mathfrak{id}$ is called a* parastrophic transformation *or* σ-parastrophic transformation, *if it can be obtained by replacing the main operation with its σ^{-1}-parastrophe.*

Two identities are called:

1) *equivalent*, if they define the same variety;

2) *primarily equivalent*, if one of them can be obtained from the other in a finite number of applications of primary identities (1)–(2) (primary equivalent identities are equivalent);

3) *σ-parastrophic*, if one of them can be obtained from the other by σ-parastrophic transformation;

4) *σ-parastrophically equivalent*, if they define σ-parastrophic varieties (according to Theorem 1., σ-parastrophically equivalent identities define σ-parastrophic varieties);

5) *σ-parastrophically primarily equivalent*, if one of them can be obtained in a finite number of applications of primary identities and σ_1 -, σ_2 -, …, σ_k - parastrophic transformations such that $\sigma_1\sigma_2\ldots\sigma_k = \sigma$ for some $k \in \mathbb{N}$.

In a generalized case σ will be omitted. For example, two identities are called *parastrophically equivalent*, if they are σ-parastrophically equivalent for some $\sigma \in S_3$.

Canonical decompositions of group isotopes

Definition 2. *A groupoid $(Q;\cdot)$ is called an* isotope of a groupoid $(Q';+)$ *iff there exists a triple of bijections (δ,ν,γ) called an* isotopism *such that $x \cdot y := \gamma(\delta^{-1}x + \nu^{-1}y)$ for all $x, y \in Q$. An isotope of a group is called a* group isotope. *A permutation α of a set Q is called unitary of a group $(Q;+,0)$, if $\alpha(0) = 0$.*

Definition 3. [5] *Let $(Q;\cdot)$ be a group isotope and 0 be an arbitrary element of Q, then the right part of the formula*

$$x \cdot y = \alpha x + a + \beta y \tag{3}$$

is called a 0-canonical decomposition*, if* $(Q;+)$ *is a group,* 0 *is its neutral element and* $\alpha 0 = \beta 0 = 0$.

In this case: the element 0 is defining element*;* $(Q;+)$ decomposition group*;* a *its* free member*;* α *its* left *or 2-coefficient;* β *its* right *or 1-coefficient;* $\alpha\beta^{-1}$ *is its* middle *or 3-coefficient. Briefly the canonical decomposition will be denoted by* $(+,0,\alpha,\beta,a)$.

Theorem 2. *[5] An arbitrary element* b *uniquely defines* b*-canonical decomposition of a group isotope.*

Let (3) be a canonical decomposition of an isotope $(Q;\cdot)$ of an Abelian group $(Q;+)$. Then it is easy to varify that all its parastrophes have the following forms

$$\begin{array}{ll} x \overset{\iota}{\cdot} y := \alpha x + a + \beta y; & x \overset{s}{\cdot} y = \beta x + a + \alpha y; \\ x \overset{\ell}{\cdot} y = \alpha^{-1}(x - a - \beta y); & x \overset{s\ell}{\cdot} y = \alpha^{-1}(-\beta x - a + y); \\ x \overset{r}{\cdot} y = \beta^{-1}(-\alpha x - a + y); & x \overset{sr}{\cdot} y = \beta^{-1}(x - a - \alpha y). \end{array} \tag{4}$$

Corollary 1. [4] *If a group isotope* $(Q;\cdot)$ *satisfies the identity*

$$w_1(x)\cdot w_2(y) = w_3(y)\cdot w_4(x) \tag{5}$$

and the variables x, y *are quadratic, then* $(Q;\cdot)$ *is isotopic to a commutative group.*

Recall, a variable is *quadratic* in an identity, if it has exactly two appearances in it. An identity is called *quadratic*, if all its variables are quadratic.

From [4, Theorem 1] follows a statement:

Theorem 3. *Let four quasigroups satisfy a quadratic identity in three variables with properties: an arbitrary sub-term of the length two (three) has two (three) different variables; any two different sub-terms of the length two have three different variables. Then all these quasigroups are isotopic to the same group.*

In the article, '*length of a term*' means the number of its individual variables including their repetitions.

Remark 1. *Let* $(Q;+,0)$ *be a group,* $Jx := -x$ *its* inverse-valued operation*,* θ *its automorphism and* φ *be its permutation. Then* $J_\varphi := \varphi^{-1}J\varphi$ *is the* inverse-valued operation *and* $\theta_\varphi := \varphi^{-1}\theta\varphi$ *is an automorphism of* φ^{-1}*-isomorphe which is denoted by* $(Q;\underset{\varphi}{+})$ *and defined by*

$$x \underset{\varphi}{+} y := \varphi^{-1}(\varphi x + \varphi y). \tag{6}$$

φ^{-1}*-isomorphe of* $(Q;+,0)$ *has the same neutral element 0, if* $\varphi 0 = 0$. *Also* $\theta_\theta = \theta$ *is true.*

Proposition 1. *A triple* (α,β,γ) *of permutations of a set* Q *is an autotopism of a commutative group* $(Q,+)$ *if and only if there exists an automorphism* θ *of* $(Q,+)$ *and elements* $b,c\in Q$ *such that*

$$\alpha = L_{c-b}\theta, \quad \beta = L_b\theta, \quad \gamma = L_c\theta.$$

Unitary component of an autotopism of an Abelian group is its automorphism.

2. Linearity of isotopes of Abelian groups

The concepts of left and right linearity are well known and were under consideration by many authors (see in [1, 7] and their cites). Following [4], we introduce these concepts basing on the concept of the canonical decomposition (which always exists and is unique) and add a new concept of middle linearity to compliment to a set of parastrophically closed notions.

Definition 4. *An isotope of an Abelian group is called*

- i-linear*, if* i*-coefficient of its canonical decomposition is an automorphism of the decomposition group,* $i = 1, 2, 3$*;*
- one-sided linear*, if it is* i*-linear for some* $i = 1, 2, 3$*;*
- linear*, if it is* i*-linear for all* $i = 1, 2, 3$*.*

Proposition 2. *The definitions of left, right and middle linearity given above are correct.*

In other words, they do not depend on canonical decompositions of the corresponding group isotopes.

Proof. Let $(+, 0, \alpha, \beta, a)$ and $(\circ, e, \alpha_1, \beta_1, b)$ be canonical decompositions of a group isotope $(Q; \cdot)$. A dependence between them is the following (see [5]):

$$x \circ y = x - e + y,\ \alpha_1 x = \alpha x - \alpha e + e,\ \beta_1 y = e - \beta e + \beta y,\ b = \alpha e + a + \beta e. \tag{7}$$

To prove independence of the left linearity concept on canonical decompositions, we have to prove that α_1 is an automorphism of the group $(Q; \circ, e)$ if and only if α is an automorphism of the group $(Q; +, 0)$. The statement 'α_1 is an automorphism of $(Q; \circ, e)$' means that for all $x, y \in Q$

$$\alpha_1(x \circ y) = \alpha_1 x \circ \alpha_1 y$$

holds. According to (7), it is equivalent to

$$\alpha(x - e + y) - \alpha e + e = \alpha x - \alpha e + e - e + \alpha y - \alpha e + e,$$

that is

$$\alpha(x - e + y) = \alpha x - \alpha e + \alpha y.$$

Replacing x with $x + e$ we have

$$\alpha(x + y) = \alpha(x + e) - \alpha e + \alpha y. \tag{8}$$

Putting $y = 0$ because $\alpha 0 = 0$, we obtain a relationship $\alpha x = \alpha(x + e) - \alpha e$. It is always true when α is an automorphism of $(Q; +, 0)$. Therefore, (8) is equivalent to $\alpha(x + y) = \alpha x + \alpha y$.

Independence of right and middle linearity concepts from canonical decompositions can be proved in the same way. □

Lemma 1. *Any two from left, right and middle linearities of an Abelian group isotope imply the third one.*

Proof. It is evident because automorphy of any two coefficients of a canonical decomposition implies automorphy of the third one. □

Lemma 2. *Let* 0 *be an arbitrary element of* Q *and* $(+,0,\alpha,\beta,a)$ *be a canonical decomposition of a commutative group isotope* $(Q;\cdot)$, *then* 0*-canonical decompositions of parastrophes of* $(Q;\cdot)$ *are the following*

$$x\overset{\iota}{\cdot}y=\alpha x+a+\beta y; \tag{9}$$

$$x\overset{s}{\cdot}y=\beta x+a+\alpha y; \tag{10}$$

$$x\overset{\ell}{\cdot}y=\alpha^{-1}x\underset{\alpha}{+}J_\alpha\alpha^{-1}a\underset{\alpha}{+}J_\alpha\beta_\alpha\alpha^{-1}y; \tag{11}$$

$$x\overset{s\ell}{\cdot}y=J_\alpha\beta_\alpha\alpha^{-1}x\underset{\alpha}{+}J_\alpha\alpha^{-1}a\underset{\alpha}{+}\alpha^{-1}y; \tag{12}$$

$$x\overset{r}{\cdot}y=J_\beta\alpha_\beta\beta^{-1}x\underset{\beta}{+}J_\beta\beta^{-1}a\underset{\beta}{+}\beta^{-1}y; \tag{13}$$

$$x\overset{sr}{\cdot}y=\beta^{-1}x\underset{\beta}{+}J_\beta\beta^{-1}a\underset{\beta}{+}J_\beta\alpha_\beta\beta^{-1}y. \tag{14}$$

Proof. If (9) is the 0-canonical decomposition of $(Q;\cdot)$, then (10) is 0-canonical decomposition of s-parastrophe of $(Q;\cdot)$.

First, we show that (11)–(14) are true. Consider (11) and (13):

$$\alpha^{-1}x\underset{\alpha}{+}J_\alpha\alpha^{-1}a\underset{\alpha}{+}J_\alpha\beta_\alpha\alpha^{-1}y\overset{(6)}{=}\alpha^{-1}(\alpha\alpha^{-1}x+\alpha J_\alpha\alpha^{-1}a+\alpha J_\alpha\beta_\alpha\alpha^{-1}y)=$$
$$=\alpha^{-1}(x+Ja+J\beta y)=\alpha^{-1}(x-a-\beta y)\overset{(4)}{=}x\overset{\ell}{\cdot}y;$$

$$J_\beta\alpha_\beta\beta^{-1}x\underset{\beta}{+}J_\beta\beta^{-1}a\underset{\beta}{+}\beta^{-1}y\overset{(6)}{=}\beta^{-1}(\beta J_\beta\alpha_\beta\beta^{-1}x+\beta J_\beta\beta^{-1}a+\beta\beta^{-1}y)=$$
$$=\beta^{-1}(J\alpha x+Ja+y)=\beta^{-1}(-\alpha x-a+y)\overset{(4)}{=}x\overset{r}{\cdot}y.$$

Thus, (11) and (13) are true.

Because $(\overset{s\ell}{\cdot})$ and $(\overset{sr}{\cdot})$ are s-parastrophes of $(\overset{\ell}{\cdot})$ and $(\overset{r}{\cdot})$ respectively, (12) and (14) hold.

Since $\alpha 0=\beta 0=0$ and $J_\alpha 0=J_\beta 0=J0=0$, all coefficients in (9)–(14) are unitary (see Remark 1.). Therefore, each of (9)–(14) is 0-canonical decomposition of the corresponding parastrophe of $(Q;\cdot)$. □

Theorem 4. *Let* $i\in\{1,2,3\}$ *and* $\sigma\in S_3$. *If an isotope of an Abelian group is* i*-linear, then its* σ*-parastrophe is* $i\sigma^{-1}$*-linear.*

Proof. Let $(Q;\cdot)$ be an isotope of an Abelian group and let (9) be its 0-canonical decomposition. According to Lemma 2., (9)–(14) imply the following table for components of 0-canonical decompositions of the parastrophes of $(Q;\cdot)$:

	ι	s	ℓ	$s\ell$	r	sr
Left coefficient	α	β	α^{-1}	$J_\alpha\beta_\alpha\alpha^{-1}$	$J_\beta\alpha_\beta\beta^{-1}$	β^{-1}
Right coefficient	β	α	$J_\alpha\beta_\alpha\alpha^{-1}$	α^{-1}	β^{-1}	$J_\beta\alpha_\beta\beta^{-1}$
Middle coefficient	$\alpha\beta^{-1}$	$\beta\alpha^{-1}$	$\beta_\alpha^{-1}J_\alpha$	$J_\alpha\beta_\alpha$	$J_\beta\alpha_\beta$	$\alpha_\beta^{-1}J_\beta$
Decomposition group	$(+)$	$(+)$	$(\underset{\alpha}{+})$	$(\underset{\alpha}{+})$	$(\underset{\beta}{+})$	$(\underset{\beta}{+})$

Let $(Q;\cdot)$ be 2-linear, i.e., left linear that is α is an automorphism of $(Q;+)$. According to Remark 1., $\alpha_\alpha = \alpha$, so, α is an automorphism of the group $(Q;\underset{\alpha}{+})$. Therefore, ℓ-parastrophe of $(Q;\cdot)$ is 2-linear, s- and $s\ell$-parastrophes are 1-linear.

Since J, α and α^{-1} are automorphisms of $(Q;+)$, Remark 1. implies that $J_\beta\alpha_\beta$ and $\alpha_\beta^{-1}J_\beta$ are automorphisms of $(Q;\underset{\beta}{+})$. It means that r- and sr-parastrophes of $(Q;\cdot)$ are 3-linear.

If $(Q;\cdot)$ is 1-linear, one can establish linearity of its parastrophes analogically.

Let $(Q;\cdot)$ be 3-linear. It means that $\alpha\beta^{-1}$ is an automorphism of $(Q;+)$. Since $(\beta\alpha^{-1})^{-1} = \alpha\beta^{-1}$, s-parastrophe of $(Q;\cdot)$ is 3-linear as well.

We claim that $J_\alpha\beta_\alpha\alpha^{-1}$ and $J_\beta\alpha_\beta\beta^{-1}$ are automorphisms of $(Q;\underset{\alpha}{+})$ and $(Q;\underset{\beta}{+})$ respectively. Indeed, according to Remark 1., J_α and $(\beta_\alpha\alpha^{-1})^{-1} = \alpha\beta_\alpha^{-1}$ are automorphisms of $(Q;\underset{\alpha}{+})$, J_β and $\alpha_\beta\beta^{-1}$ are automorphisms of $(Q;\underset{\beta}{+})$.

Therefore, ℓ- and sr-parastrophes of $(Q;\cdot)$ are 2-linear, $s\ell$- and r-parastrophes are 1-linear. □

Using the concepts left, right and middle linearity, the theorem immediately implies

Corollary 2. *Relationships among one-sided linearity of an isotope of an Abelian group and its parastrophes are given in the following table:*

Linearity	ι	ℓ	r	s	$s\ell$	sr
left	*left*	*left*	*middle*	*right*	*right*	*middle*
right	*right*	*middle*	*right*	*left*	*middle*	*left*
middle	*middle*	*right*	*left*	*middle*	*left*	*right*

Definition 5. *A quasigroup $(Q;\cdot)$ will be called* strictly i-linear, *if it is i-linear and it is not j-linear for some $j \neq i$.*

For example, if a quasigroup is 1-linear, then the right coefficient is an automorphism of the decomposition group and neither left coefficient, nor middle coefficient is an automorphism. It follows from Lemma 1..

Example 1. *The quasigroup $(\mathbb{Z}_4;\circ)$ which is defined by $x \circ y := (13)x + (23)y$ is strictly left central, since the circle (13) is an automorphism of $(\mathbb{Z}_4;+)$ and neither the circle (23), nor the circle $(13)(23) = (123)$ is an automorphism of $(\mathbb{Z}_4;+)$. Therefore, its parastrophes $(\mathbb{Z}_4;\overset{s}{\circ})$ and $(\mathbb{Z}_4;\overset{r}{\circ})$ are strictly right central and middle central respectively.*

3. Linearity of varieties and corresponding identities

Definition 6. *A class of quasigroups will be called*

- i-linear, *if all its quasigroups are i-linear;*
- involutional i-linear, *if all its quasigroups are i-linear and their canonical i-th coefficients are involutions;*
- strictly i-linear, *if all its quasigroups are i-linear and there exists a strictly i-linear quasigroup in the class.*

- i-central, *if all its quasigroups are i-central, i.e., their canonical decomposition groups are commutative;*
- strictly i-central, *if all its quasigroups are i-central and there exists a strictly i-central quasigroup in the class.*

Consequently, the *involutional strictly left central quasigroup variety* is a variety of left linear quasigroups which are isotopic to commutative groups and left coefficients in their canonical decompositions are involutions, i.e., the functions that are their own inverse.

Lemma 3. *Any two from 1-, 2-, 3-linearities of classes of quasigroups imply the third one.*

Proof. It immediately follows from Definition 6. and Lemma 1.. □

Definition 7. *An identity is called* i-linear *(*strictly i-linear*), if it defines i-linear (strictly i-linear) variety of quasigroups.*

Theorem 5. *Let $i \in \{1,2,3\}$ and $\sigma \in S_3$. If a class of quasigroups is i-linear, then its σ-parastrophe is $i\sigma^{-1}$-linear.*

Proof. Let a class of quasigroups $\mathfrak{A}$ be i-linear. It means that each quasigroup $(Q;\cdot)$ from $\mathfrak{A}$ is i-linear. Then according to Theorem 4., σ-parastrophe of $(Q;\cdot)$ is $i\sigma^{-1}$-linear. Since ${}^{\sigma}\mathfrak{A}$ consists of all σ-parastrophes of quasigroups from $\mathfrak{A}$, the class ${}^{\sigma}\mathfrak{A}$ is $i\sigma^{-1}$-linear. □

Theorem 6. *Let $i \in \{1,2,3\}$ and $\sigma \in S_3$. If a class of quasigroups is strictly i-linear, then its σ-parastrophe is strictly $i\sigma^{-1}$-linear.*

Proof. Let a class of quasigroups $\mathfrak{A}$ be strictly i-linear, i.e., all quasigroups from $\mathfrak{A}$ are i-linear and there exists a quasigroup $(Q;\cdot)$ in $\mathfrak{A}$ which is not j-linear for some $j \neq i$. According to Theorem 5. all quasigroups from ${}^{\sigma}\mathfrak{A}$ are $i\sigma^{-1}$-linear. The σ-parastrophe $(Q;\overset{\sigma}{\cdot})$ is not $j\sigma^{-1}$-linear, otherwise σ^{-1}-parastrophe of $(Q;\overset{\sigma}{\cdot})$, i.e. the quasigroup

$$(Q;{}^{\sigma^{-1}}(\overset{\sigma}{\cdot})) = (Q;\cdot)$$

is j-linear. Therefore, the quasigroup $(Q;\overset{\sigma}{\cdot})$ and consequently the variety ${}^{\sigma}\mathfrak{A}$ are strictly $i\sigma^{-1}$-linear. □

According to Definition 7., Theorems 5., 6. imply the following

Corollary 3. *Let $i \in \{1,2,3\}$ and $\sigma \in S_3$. If a class of quasigroups is i-linear (strictly i-linear), then its σ-parastrophe is $i\sigma^{-1}$-linear (strictly $i\sigma^{-1}$-linear).*

Further we will consider varieties $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ defined by identities

$$(xy \cdot u)v = (xv \cdot u)y, \tag{15}$$

$$x(y \cdot uv) = u(y \cdot xv), \tag{16}$$

$$x(y \overset{r}{\cdot} uv) = u(vx \overset{\ell}{\cdot} y) \tag{17}$$

respectively. The following theorem is true.

Theorem 7. *The varieties* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ *are different.*
1. $\mathfrak{A} = {}^{\ell}\mathfrak{A}$ *is the variety of all involutional left central quasigroups;*
2. $\mathfrak{B} = {}^{s}\mathfrak{A} = {}^{s\ell}\mathfrak{A}$ *is the variety of all involutional right central quasigroups;*
3. $\mathfrak{C} = {}^{r}\mathfrak{A} = {}^{sr}\mathfrak{A}$ *is the variety of all involutional middle central quasigroups.*

Proof. **1.** According to Definition 1., ℓ-parastrophe of (15) has the form

$$((x \overset{\ell}{\cdot} y) \overset{\ell}{\cdot} u) \overset{\ell}{\cdot} v = ((x \overset{\ell}{\cdot} v) \overset{\ell}{\cdot} u) \overset{\ell}{\cdot} y.$$

In the right part of the equality we use the definition of the left division:

$$(x \overset{\ell}{\cdot} v) \overset{\ell}{\cdot} u = \big(((x \overset{\ell}{\cdot} y) \overset{\ell}{\cdot} u) \overset{\ell}{\cdot} v\big)y.$$

Replace x with $xu \cdot y$ and apply the first identity from (1):

$$((xu \cdot y) \overset{\ell}{\cdot} v) \overset{\ell}{\cdot} u = (x \overset{\ell}{\cdot} v)y.$$

According to the definition of the left division, we have

$$(x \overset{\ell}{\cdot} v)y \cdot u = (xu \cdot y) \overset{\ell}{\cdot} v.$$

In the right part of the equality we apply the definition of the left division. Replacing x with xv and using the first identity from (1), we receive an identity which coincides with (15).

Thus, (15) is equivalent to its ℓ-parastrophe, i.e. $\mathfrak{A} = {}^{\ell}\mathfrak{A}$.

Let a quasigroup $(Q;\cdot)$ satisfy the identity (15). Put $x = a$ ($a \in Q$) in (15) and define the operation $(\circ)$ by

$$x \circ y := L_a x \cdot y,$$

then (15) can be written as follows

$$(y \circ u) \cdot v = (v \circ u) \cdot y.$$

The last identity satisfies conditions of Theorem 3., then $(Q;\cdot)$ is a group isotope. The identity (15) has the form (5) with respect to the quadratic variables y and v. In virtue of Corollary 1., $(Q;\cdot)$ is isotopic to some Abelian group. It implies that (3) is 0-canonical decomposition of $(Q;\cdot)$. Applying (9) to (16), we have

$$\alpha\big(\alpha(\alpha x + a + \beta y) + a + \beta u\big) + a + \beta v = \alpha\big(\alpha(\alpha x + a + \beta v) + a + \beta u\big) + a + \beta y. \tag{18}$$

Since $\alpha 0 = \beta 0 = 0$, then the last equality at $x = y = 0$ implies that α is an automorphism of $(Q;+)$.

Putting $x = u = v = 0$ in (18), we obtain $\alpha^2\beta = \beta$, that is, $\alpha^2 = \iota$. Therefore, $(Q;\cdot)$ is a left involutional central quasigroup.

Vice versa, let $(Q;\cdot)$ be a left involutional central quasigroup, i.e.,

$$x \cdot y = \alpha x + a + \beta y, \qquad \alpha^2 = \iota, \tag{19}$$

for an automorphism α of a group $(Q;+)$, a permutation β of Q with $\beta 0 = 0$ and an element $a \in Q$. We claim that $(Q;\cdot)$ satisfies (15). Indeed,

$$(xy \cdot u) \cdot v \overset{(19)}{=} \alpha\big(\alpha(\alpha x + a + \beta y) + a + \beta u\big) + a + \beta v.$$

Since α is an automorphism of $(Q;+)$, then

$$(xy \cdot u) \cdot v = \alpha^3 x + \alpha^2 a + \alpha^2 \beta y + \alpha a + \alpha\beta u + a + \beta v.$$

Because the group $(Q;+)$ is Abelian and $\alpha^2 = \iota$, we have

$$(xy \cdot u) \cdot v = \alpha^3 x + \alpha^2 a + \beta y + \alpha a + \alpha\beta u + a + \beta v =$$

$$= \alpha^3 x + \alpha^2 a + \beta v + \alpha a + \alpha\beta u + a + \beta y =$$

$$= \alpha^3 x + \alpha^2 a + \alpha^2 \beta v + \alpha a + \alpha\beta u + a + \beta y =$$

$$= \alpha\big(\alpha(\alpha x + a + \beta v) + a + \beta u\big) + a + \beta y \overset{(19)}{=} (xv \cdot u) \cdot y.$$

Thus, $(Q;\cdot)$ satisfies the identity (15) and therefore item 1. has been proved.

2. s-parastrophe of (15) is

$$((x \overset{s}{\cdot} y) \overset{s}{\cdot} u) \overset{s}{\cdot} v = ((x \overset{s}{\cdot} v) \overset{s}{\cdot} u) \overset{s}{\cdot} y.$$

By the definition of s-parastrophe, we obtain $v(u \cdot yx) = y(u \cdot vx)$. It coincides with (16) after mutual relabeling of x and v, y and u. Therefore, (16) is s-parastrophically equivalent to (15), i.e. $\mathfrak{B} = {}^s\mathfrak{A}$. Moreover,

$$\mathfrak{B} = {}^s\mathfrak{A} = {}^s({}^\ell\mathfrak{A}) = {}^{s\ell}\mathfrak{A}.$$

By the definition, a quasigroup $(Q;\cdot)$ belongs to the variety ${}^s\mathfrak{A}$ if and only if its s-parastrophe $(Q;\overset{s}{\cdot})$ belongs to the variety $\mathfrak{A}$. Let

$$x \overset{s}{\cdot} y = \alpha x + a + \beta y$$

be a canonical decomposition of $(Q;\cdot)$. $\mathfrak{A}$ is the variety of all involutional left central quasigroups. It means that $(Q;+)$ is an Abelian group, α is its automorphim and $\alpha^2 = \iota$. According to (10) of Lemma 2.,

$$x \cdot y = \beta x + a + \alpha y. \tag{20}$$

Therefore every quasigroup from ${}^s\mathfrak{A}$ is involutional right central one.

Let $(Q;\cdot)$ be an arbitrary involutional right central quasigroup. Then $(Q;\overset{s}{\cdot})$ is a involutional left central quasigroup, i.e., $(Q;\overset{s}{\cdot})$ belongs to $\mathfrak{A}$. Consequently, $(Q;\cdot)$ belongs to ${}^s\mathfrak{A}$. Thus, ${}^s\mathfrak{A}$ is the variety of all involutional right central quasigroups.

3. The following identity is r-parastrope of (15)

$$((x \overset{r}{\cdot} y) \overset{r}{\cdot} u) \overset{r}{\cdot} v = ((x \overset{r}{\cdot} v) \overset{r}{\cdot} u) \overset{r}{\cdot} y.$$

Replace y with xy, v with xv and use the third identity from (1):

$$(y \overset{r}{\cdot} u) \overset{r}{\cdot} xv = (v \overset{r}{\cdot} u) \overset{r}{\cdot} xy.$$

In the left part of the equality we apply the definition of the right division. Replacing u with vu and using the third identity from (1), we receive

$$(y \overset{r}{\cdot} vu) \cdot (u \overset{r}{\cdot} xy) = xv.$$

Put v instead of $y \overset{r}{\cdot} vu$, i.e., $yv \overset{\ell}{\cdot} u$ instead of v:

$$v(u \overset{r}{\cdot} xy) = x(yv \overset{\ell}{\cdot} u).$$

Relabeling variables according to the cycle $(xuyv)$, we obtain (17). It means that (17) and (15) are r-parastrophically primary equivalent, so, $\mathfrak{C} = {}^r\mathfrak{A}$.

Moreover, because $r\ell = sr$, we have

$$\mathfrak{C} = {}^r\mathfrak{A} = {}^r({}^\ell\mathfrak{A}) = {}^{r\ell}\mathfrak{A} = {}^{sr}\mathfrak{A}.$$

By the definition, a quasigroup $(Q;\cdot)$ belongs to the variety ${}^r\mathfrak{A}$ if and only if its r-parastrophe $(Q;\overset{r}{\cdot})$ belongs to the variety $\mathfrak{A}$. Let

$$x \cdot y = \alpha x + a + \beta y$$

be 0-canonical decomposition of $(Q;\cdot)$. According to (13) of Lemma 2., 0-canonical decomposition of $(Q;\overset{r}{\cdot})$ is

$$x \overset{r}{\cdot} y = \delta x \underset{\beta}{+} J_\beta \beta^{-1} a \underset{\beta}{+} \beta^{-1} y,$$

where $\delta := J_\beta \alpha_\beta \beta^{-1}$. By virtue of Remark 1., $J_\beta := \beta^{-1} J \beta$, $\alpha_\beta := \beta^{-1}\alpha\beta$ and

$$x \underset{\beta}{+} y := \beta^{-1}(\beta x + \beta y).$$

Therefore, $\delta = (\beta^{-1}J\beta)(\beta^{-1}\alpha\beta)\beta^{-1} = \beta^{-1}J\alpha$. Since $(Q;\overset{r}{\cdot})$ belongs to $\mathfrak{A}$, then $\delta^2 = \iota$ and δ is an automorphism of the commutative group $(Q;\underset{\beta}{+},0)$.

Consequently, $\delta(x \underset{\beta}{+} y) = \delta x \underset{\beta}{+} \delta y$, i.e.,

$$\delta\beta^{-1}(\beta x + \beta y) = \beta^{-1}(\beta\delta x + \beta\delta y),$$

Replace βx with x and βy with y: $\beta\delta\beta^{-1}(x + y) = \beta\delta\beta^{-1}x + \beta\delta\beta^{-1}y$. Therefore, $\beta\delta\beta^{-1} = \beta(\beta^{-1}J\alpha)\beta^{-1} = J\alpha\beta^{-1}$ is an automorphism of the commutative group $(Q;+)$, so, $\alpha\beta^{-1}$ is an automorphism of the group. Mereover, $\delta^2 = \iota$ means $\beta^{-1}J\alpha\beta^{-1}J\alpha = \iota$, or $J^2\alpha\beta^{-1}\alpha = \beta$, i.e. $(\alpha\beta^{-1})^2 = \iota$. Thus, $(Q;\cdot)$ is an involutional middle central quasigroup. □

Conclusions

There are three one-sided notions of centrality of quasigroups (i.e., linearity of isotopes of Abilian groups): left centrality (i.e. 2-centrality), right centrality (i.e. 1-centrality) and middle centrality (i.e. 3-centrality). A dependence among these notions is: *If a quasigroup is i-central, then its σ-parastrophe is $i\sigma^{-1}$-central.* The set of all isotopes of a commutative group is parted into three disjoint parastrophically closed subsets: non-linear quasigroups; strictly one-sided central quasigroups and central quasigroups.

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ПРО ЛІНІЙНІСТЬ ІЗОТОПІВ АБЕЛЕВИХ ГРУП

РЕЗЮМЕ

Введено поняття середньої лінійності для ізотопів абелевих груп. Якщо квазігрупа має односторонню лінійність, то серед її парастрофів: два ліволінійні, два праволінійні та два середньолінійні. Знайдена відповідна формула. А також встановлено залежність між відповідними тотожностями та многовидами. Охарактеризовано інволютивні односторонньо центральні квазігрупи.

Ключові слова: *квазігрупа, лінійна квазігрупа, груповий ізотоп.*

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О ЛИНЕЙНОСТИ ИЗОТОПОВ АБЕЛЕВЫХ ГРУПП

РЕЗЮМЕ

Введено понятие средней линейности для изотопов абелевых групп. Если квазигруппа имеет одностороннюю линейность, то среди ее парастрофов: два леволинейные, два праволинейные и два среднелинейные. Найдена соответствующая формула. А также установлена зависимость между соответствующими тождествами и многообразиями. Охарактеризованы инволютивные односторонне центральные квазигруппы.

Ключевые слова: *квазигруппа, линейная квазигруппа, групповой изотоп.*