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MINIMAL GENERATING SET AND A STRUCTURE OF THE WREATH PRODUCT OF GROUPS, AND THE FUNDAMENTAL GROUP OF THE ORBIT MORSE FUNCTION

The quotient group of the restricted and unrestricted wreath product by its commutator is found. The generic sets of commutator of wreath product were investigated.

The structure of wreath product with non-faithful group action is investigated.

Given a permutational wreath product sequence of cyclic groups, we investigate its minimal generating set, the minimal generating set for its commutator and some properties of its commutator subgroup.

We strengthen the results from the author [18, 21, 20] and construct the minimal generating set for the wreath product of both finite and infinite cyclic groups, in addition to the direct product of such groups. We generalise the results of Meldrum J. [12] about commutator subgroup of wreath products since, as well as considering regular wreath products, we consider those which are not regular (in the sense that the active group $\mathcal{A}$ does not have to act faithfully). The commutator of such a group, its minimal generating set and the center of such products has been investigated here.

The fundamental group of orbits of a Morse function $f : M \to \mathbb{R}$ defined upon a Möbius band M with respect to the right action of the group of diffeomorphisms $\mathcal{D}(M)$ has been investigated. In particular, we describe the precise algebraic structure of the group $\pi_1 O(f)$. A minimal set of generators for the group of orbits of the functions $\pi_1(O_f, f)$ arising under the action of the diffeomorphisms group stabilising the function f and stabilising ∂M have been found. The Morse function f has critical sets with one saddle point.

We consider a new class of wreath-cyclic geometrical groups. The minimal generating set for this group and for the commutator of the group are found.

Key words: wreath product of group; minimal generating set of commutator subgroup of wreath product of groups, center of non regular wreath product, semidirect product, fundamental group of orbits of one Morse function, groups of diffeomorphisms acting on the Möbius band, wreath-cyclic group

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Introduction

Lucchini A. [10] previously investigated a case of the generating set of $C_p^{n-1} \wr G$, where G denotes a finite n-generated group, p is a prime which does not divide the order $|G|$ and C_p denotes the cyclic group of order p. The results of Lucchini A. [10] tell us that the wreath product $C_p^{n-1} \wr G$ is also n-generated. We firstly consider the active group G which is cyclic and then generalise this wreath product for both iterated wreath products and for the direct product of wreath products of cyclic groups. It should be noted that a similar question for iterated wreath product was studied by Bondarenko I. [3].

Maksymenko S. [11] has proven that the n-th homotopy groups of the orbit $O(f)$, of f, with respect to the right action of the group, $\mathrm{Diff}(M)$, of diffeomorphisms of M, coincides with those of M for $n \geqslant 3$, i.e. $\pi_2 O(f) = 0$, while, for the fundamental group $\pi_1 O(f)$, it is known that it contains a free abelian subgroup of finite index. Despite this, information regarding the fundamental group $\pi_1 O(f)$ remains incomplete. We provide some insight by finding the minimal generating set and its relations for the group $\pi_1 O(f)$.

Preliminaries

We denote by $d(G)$ the minimal number of generators of the group G [10, 3]. A diffeomorphism $h : M \to M$ is said to be f-preserving if $f \circ h = f$. This is equivalent to the assumption that h is invariant each level-set, i.e. $f^{-1}(c)$, $c \in P$ of f, where P denotes either the real line R or the circle S^1.

Let G be a group. The *commutator width* of G [15], denoted $cw(G)$, is defined to be the least integer n, such that every element of G' is a product of at most n commutators if such an integer exists, and otherwise is $cw(G) = \infty$. The first example of a finite perfect group with $cw(G) > 1$ was presented by Isaacs I. [7]. The property of commutator widths for groups and elements has proven to be important and in particular, its connections with stable commutator length and bounded cohomology has become significant.

Meldrum J. [12] briefly considered one form of commutators of the wreath product $A \wr B$. In order to obtain a more detailed description of this form, we take into account the commutator width $(cw(G))$ as presented in work of Muranov A. [13].

The form of commutator presentation [12] has been given here in the form of wreath recursion [9] and additionally, its commutator width has been studied.

The subtree of X^* (or $\mathbb{T}$) which is induced by the set of vertices $\cup_{i=0}^{k} X^i$ is denoted by $X^{[k]}$ (or $\mathbb{T}_k$). Denote the restriction of the action of an automorphism $g \in AutX^*$ to the subtree $X^{[l]}$ by $g_{(v)}|_{X^{[l]}}$. It should be noted that a restriction $g_{(v)}|_{X^{[1]}}$ is called the *vertex permutation* (v.p) of g in a vertex v.

Center and commutator subgroup of wreath product

This work strengthens previous results by the author [18] and will additionally consider a new class of groups. This class is precisely the *wreath-cyclic* groups and will be denoted by $\Im$. Let $G \in \Im$, then this class is constructed by formula:

$$G = (\wr_{j_0=0}^{n_0} C_{k_{j_0}}) \times (\wr_{j_1=0}^{n_1} C_{k_{j_1}}) \times \cdots \times (\wr_{j_l=0}^{n_l} C_{k_{j_l}}), 1 \leqslant k_{j_i} < \infty, n_i < \infty,$$

where the orders of C_{i_j} are denoted by i_j.

It should be noted that at the end of this product, a semidirect product could arise with a given homomorphism ϕ, which is defined by a free action on the set $\mathbb{Z}$. In other words, one would obtain a group of the form $\left(\prod_{i=1}^{k} G_i\right)^n \ltimes_\phi \mathbb{Z}$.

Note that the last group here is isomorphic to one of the fundamental orbital groups $O_f(f)$ of the Morse function f. Namely, we have $\pi_0(S, f|_{\partial M})$ [11].

Consider now the group $H = \wr_{j=1}^{n} C_{i_j}$, whose orders i_j for all C_{i_j} are mutually coprime for all $j > 1$ and whose number of cyclic factors in the wreath product is finite. We will call such group H *wreath-cyclic*.

Note that the multiplication rule of automorphisms g, h which are presented in the form

of wreath recursion [14]

$$g = (g_{(1)}, g_{(2)}, \ldots, g_{(d)})\sigma_g,\ h = (h_{(1)}, h_{(2)}, \ldots, h_{(d)})\sigma_h,$$

is given precisely by the formula:

$$g \cdot h = (g_1 h_{\sigma_g(1)}, g_2 h_{\sigma_g(2)}, \ldots, g_d h_{\sigma_g(d)})\sigma_g\sigma_h.$$

In the general case, if an active group is not cyclic, then the cycle decomposition of an n-tuple for automorphism sections will induce the corresponding decomposition of the σ_g. If σ is v.p of automorphism g at v_{ij} and all the vertex permutations below v_{ij} are trivial, then we do not distinguish σ from the section $g_{v_{ij}}$ of g which is defined by it. That is to say, we can write $g_{v_{ij}} = \sigma = (v_{ij})g$ as proposed by Bartholdi L., Grigorchuk R. and Šuni Z. [1].

Minimal generating set of direct product of wreath product of cyclic groups

We now make use of both rooted and directed automorphisms as introduced by Bartholdi L., Grigorchuk R. and Šuni Z. [1]. Recall that we denote a truncated tree by $\mathbb{T}$.

Definition 1. *An automorphism of $\mathbb{T}$ is said to be rooted if all of its vertex permutations corresponding to non-empty words are trivial.*

Let $l = x_1x_2x_3\cdots$ be an infinite ray in $\mathbb{T}$.

Definition 2. *The automorphism g of $\mathbb{T}$ is said to be directed along the infinite ray l if all vertex permutations along l and all vertex permutations corresponding to vertices whose distance to the ray l is at least two are trivial. In such case, we say that l is the spine of g (as exemplified in Figure 1).*

It should be noted that because we consider only truncated trees and truncated automorphisms here and for convenience, we will say rooted automorphism instead of truncated rooted automorphism.

Theorem 1. *If orders of cyclic groups $\mathbb{C}_{n_i}$, $\mathbb{C}_{n_j}$ are mutually coprime $i \neq j$, then the group $G = C_{i_1} \wr C_{i_2} \wr \cdots \wr C_{i_m}$ admits two generators, namely β_0, β_1.*

Proof. Construct the generators of $\wr_{j=0}^{n} C_{i_j}$ as a rooted automorphism β_0 (Figure 2) and a directed automorphism β_1 [1] along a path l (Figure 1) on a rooted labeled truncated tree T_X.

We consider the group $G = C_{i_1} \wr C_{i_2} \wr \cdots \wr C_{i_m}$. Construct the generating set of $C_{i_1} \wr C_{i_2} \wr \cdots \wr C_{i_m}$, where the active group is on the left. Denote by $lcm_1 = lcm(i_2, i_3, \ldots, i_m)$ the least common multiplier of the orders by $i_2, i_3, \ldots, i_m$. In a similar fashion, we denote

$$lcm_k = lcm(i_1, i_2, \ldots, i_{k-1}, i_{k+1}, \ldots, i_m)$$

similarly.

We utilise a presentation of those wreath product elements from a tableaux of Kaloujnine L. [8] which has the form $\sigma = [a_1, a_2(x),\ a_3(x_1, x_2), \ldots]$. Additionally, we use a subgroup of tableau with length n which has the form $\sigma_{(n)} = [a_1, a_2(x_1), \ldots a_n(x_1, \ldots, x_n)]$. The tableaux which has first n trivial coordinates was denoted in [23] by

$${}^{(n)}\sigma = [e, \ldots, e, \alpha_{n+1}(x_1, \ldots, x_n), \alpha_{n+1}(x_1, \ldots, x_{n+1}), \ldots].$$

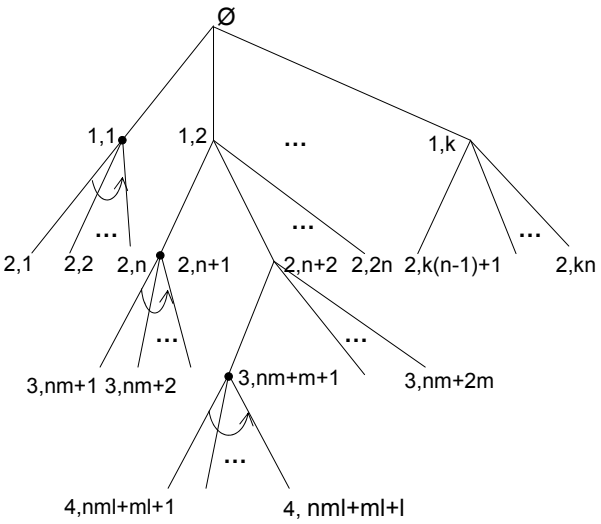

Fig. 1. Directed automorhism

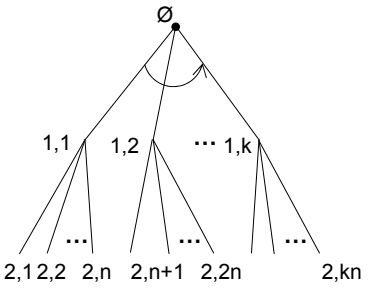

Fig. 2. Rooted automorhism

The canonical set of generators for the wreath product of $C_p \wr \cdots \wr C_p \wr C_p$ was used by Dmitruk Y. and Sushchanskii V. [5] and additionally utilised by the author [17]. This set has form

$$\sigma'_1 = [\pi_1,\, e,\, e, \ldots, e]\,, \sigma'_2 = [e_1,\, \pi_2,\, e, \ldots, e]\,, ..., \sigma'_n = [e_1,\, e, \ldots, e, \pi_n]\,. \tag{1}$$

We split such a table into sections with respect to (1), where the i-th section corresponds to portrait of α at i-th level. The first section corresponds to an active group and the crown of wreath product G, the second section is separated with a semicolon to a base of the wreath product. The sections of the base of wreath product are divided into parts by semicolon and these parts correspond to groups C_{i_j} which form the base of wreath product. The l-th section of of a tableau presentation of automorphism β_1 corresponds to portrait of automorphism β_1 on level X^l.

The portrait of automorphisms β_1 on level X^l is characterised by the sequence $(e, \ldots, e, \pi_l, e, \ldots, e)$, where coordinate π_l is the vertex number of unique non trivial v.p on X^l, the sequence has $i_0 i_1 \ldots i_{l-1}$ coordinates. Therefore, our first generator has the form $\beta_0 = [\pi_1, e, e, \ldots, e]$, which is the rooted automorphism. The second generator has the form

$$\beta_1 = \left[e; \underbrace{\pi_2,\, e, e, \ldots, e}_{i_1}; \overbrace{\underbrace{e, e, \ldots, e}_{i_2}, \pi_3, e, \ldots, e}^{i_1 i_2}; \overbrace{\underbrace{e, \ldots, e}_{i_1 i_2 i_3}, \pi_4, e, \ldots, e}^{i_2 i_3 + i_3}; e, \ \ldots \ , e \right],$$

It should be noted that after the last(fourth) semicolon (or in other words before π_5) there are $i_2 i_3 i_4 + i_3 i_4 + i_4$ trivial coordinates. There are $i_2 i_3 i_4 i_5 + i_3 i_4 i_5 + i_4 i_5 + i_5$ trivial coordinates before π_6 (or in other words after the fifth semicolon but before π_6). In a section after $k-1$ semicolon the coordinate of a non-trivial element π_k is calculated in a similar way. We know from [23] that β_1 is generator of ${}^{(2)}G$, i.e. 2-base of G. Recall that ${}^{(k)}G$ calls k-th base of G. The subgroup ${}^{(k)}G$ is a subgroup of all tableaux of form ${}^{(k)}u$ with $u \in G$.

Let $C_n = \langle \pi_n \rangle$ and set $\sigma_1 = \beta_0$. We have to show that our generating set $\{\beta_0, \beta_1\}$ generates the whole canonical generating set. For this, we obtain the second new generator σ_2

in form of the tableau

$$\sigma_2^{lcm_2} = \beta_1^{lcm_2} = \left[e; \underbrace{\pi_2^{lcm_2}, e, e, \ldots, e}_{i_1}; \underbrace{e, e, \ldots, e}_{i_1 i_2}; \underbrace{e, e, \ldots, e}_{i_1 i_2 i_3}; e, \ldots, e \right].$$

Because $\mathrm{ord}(\pi_1) = i_1$ and $(i_1,\ lcm_1) = 1$, we find that the element $\pi_1^{lcm_1}$ is generator of C_{i_1} since $\mathrm{ord}(\pi_1) = \mathrm{ord}(\pi_1^{lcm_1})$. We obtain that

$$\sigma_2 = \left(\beta_1^{lcm_2}\right)^{lcm_2^{-1}(mod\, i_2)},$$

which corresponds to generator σ_2 of canonical generating set (1). Observe that $b_3 = \sigma_1^{-1}\beta_1$ is generator of ${}^{(3)}G$, i.e. it is precisely a 3-base of G.

It is known [23] that the generator σ_2 precisely generates the group that is isomorphic to the group $[U]_2$ for all 2-nd coordinate tableaux. From the same principle, one can obtain that

$$\sigma_3 = \beta_1^{lcm_3} = \left[e; \underbrace{e,\ e, \ldots, e}_{i_1}; \overbrace{\underbrace{e, e, \ldots, e}_{i_2}, \pi_3^{lcm_3}, e, \ldots, e}^{i_1 i_2}; \overbrace{\underbrace{e, \ldots, e}_{i_1 i_2 i_3}, e, \ldots, e}^{i_2 i_3}; e\ \ldots\ e \right].$$

This generator σ_3 generates the group which is isomorphic to the group of all $(2i_1 + 2)$-th coordinate tableaux, which is precisely $[U]_{2i_1+2}$ [23]. Making use of the same principle allows us to express all the σ_i from our canonical generating set.

Note that if it were a self-similar group, then it would be more useful to present it in terms of wreath recursion, as the set where β_0 is the rooted automorphism. Given a permutational representation of C_{i_j} we can present our group by wreath recursion. We present β_1 by wreath recursion as $\beta_1 = (\pi_2, \beta_2, e, e, \ldots, e)$. It would be written in form $\sigma_1^{lcm_2} = \beta_1{}^{lcm_2} = (\pi_2{}^{lcm_2}, \beta_2{}^{lcm_2}, e, e, \ldots, e) = (\pi_2{}^{lcm(2)}, e, e, \ldots, e)$, since $\mathrm{ord}(\pi_2) = i_2$ and $(i_2,\ lcm_2) = 1$ then the element $\pi_2^{lcm_2}$ is generator of C_{i_2} too, because $\mathrm{ord}(\pi_2) = \mathrm{ord}(\pi_2^{lcm_2})$.

We then obtain the second generator σ_2 of canonical generating set by exponentiation $\left(\beta_1^{lcm_2}\right)^{lcm_2^{-1}(mod\, i_2)} = (\pi_2, e, \ldots, e)$. Since we have obtained $\sigma_2 = (\pi_2, e, \ldots, e)$, we can express $\sigma_2^{-1} = \left(\pi_2^{-1}, e, \ldots, e\right)$, where π_2 is a state of σ_2.

Consider an alternative recursive constructed generating set which consists of nested automorphism β_1 states which are β_2, β_3 ,$\ldots, \beta_m$ and the automorphism β_0. The state β_2 is expressed as follows $\sigma_2^{-1}\beta_1 = (e, \beta_2, e, \ldots, e)$.

It should be noted that a second generator of a recursive generating set could be constructed in an other way, namely $\beta'_2 = \beta_1{}^{i_2} = (\pi_2{}^{i_2}, \beta_2{}^{i_2}, e, e, \ldots, e) = (e, \beta_2{}^{i_2}, \ldots, e, e)$, where β_2 is the state in a vertex of the second level X^2.

We can then express the next state β_2 of β_1 by multiplying $\sigma_2^{-1}\beta_1 = (e, \beta_2, e, \ldots, e)$. Therefore, by a recursive approach, we obtain $\beta_2 = (\pi_3, \beta_3, e, ..., e)$ and analogously we obtain $\beta_2^{lcm_3} = \sigma_3^{lcm_3} = (\pi_3^{lcm_3}, e, \ldots, e)$. Similarly, we obtain

$$\beta_{k-1}^{lcm_k} = \sigma_k^{lcm_k} = \left(\pi_k^{lcm_k}, e, \ldots, e\right)$$

and $\sigma_k = \left(\beta_{k-1}^{lcm_k}\right)^{lcm_k^{-1}(\mathrm{mod}\ i_k)} = (\pi_k, e, \ldots, e)$. The k-th generator of the recursive generating set can therefore be expressed as $\sigma_k^{-1}\beta_{k-1} = (e, \beta_k, e, \ldots, e)$.

The last generator of our generating set has another structure, namely $\sigma_m = (\pi_m, e, \ldots, e)$ which concludes the proof. □

Let $\wr_{j=0}^{n} C_{i_j}$ be generated by β_0 and β_1 and $\wr_{l=0}^{m} C_{k_l} = \langle \alpha_0, \alpha_1 \rangle$. Denote an order of g by $|g|$.

Theorem 2. *If* $(|\alpha_0|, |\beta_0|) = 1$ *and* $(|\alpha_1|, |\beta_1|) = 1$ *or* $(|\alpha_0|, |\beta_1|) = 1$ *and* $(|\alpha_1|, |\beta_0|) = 1$, *then there exists generating set of 2 elements for the wreath-cyclic group*

$$G = (\wr_{j=0}^{n} C_{i_j}) \times (\wr_{l=0}^{m} C_{k_l}),$$

where i_j *are orders of* C_{i_j}.

Proof. The generators α_1 and β_1 are directed automorphisms, α_0, β_0 are rooted automorphisms [1]. The structure of tableaux are described above in Theorem 1. In case $(|\alpha_0|, |\beta_0|) = 1$ are mutually coprime and $(|\alpha_1|, |\beta_1|) = 1$ are mutually coprime, then we group generator α_0 and β_0 in vector that is first generator of direct product $(\wr_{j=0}^{n} C_{i_j}) \times (\wr_{l=0}^{m} C_{k_l})$. Therefore, the first generator of G has form $(\alpha_0,\ \beta_0)$ and the second generator has form of vector $(\beta_1,\ \alpha_1)$. The generator α_1 has a similar structure.

In order to express the generator σ_2 of the canonical set (1) from $\langle \alpha_0, \beta_1 \rangle$ we change the exponent from β_1 to lcm_2. Analogously, we obtain $\sigma_k = \beta_1{}^{lcm_k}$ which concludes the proof. □

Generators of commutator and center of wreath product

The following Lemma imposes the Corollary 4.9 of [12].

Lemma 1. *An element of form* $(r_1, \ldots, r_{p-1}, r_p) \in W' = (B \wr C_p)'$ *iff product of all* r_i *(in any order) belongs to* B', *where* $p \in \mathrm{N}$, $p \geqslant 2$.

Proof. Analogously to the Corollary 4.9 of the Meldrum's book [12] we can deduce new presentation of commutators in form of wreath recursion

$$w = (r_1, r_2, \ldots, r_{p-1}, r_p),$$

where $r_i \in B$. If we multiply elements from a tuple $(r_1, \ldots, r_{p-1}, r_p)$, where $r_i = h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)}$, $h,\ g \in B$ and $a, b \in C_p$, then we get a product

$$x = \prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} \in B', \tag{2}$$

where x is a product of corespondent commutators. Therefore, we can write $r_p = r^{-1}_{p-1} \ldots r^{-1}_1 x$. We can rewrite element $x \in B'$ as the product $x = \prod_{j=1}^{m} [f_j, g_j]$, $m \leqslant cw(B)$.

Note that we impose more weak condition on the product of all r_i to belongs to B' then in Definition 4.5. of form $P(L)$ in [12], where the product of all r_i belongs to a subgroup L of B such that $L > B'$.

In more detail deducing of our representation constructing can be reported in following way. If we multiply elements having form of a tuple $(r_1, \ldots, r_{p-1}, r_p)$, where

$r_i = h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)}$, $h, g \in B$ and $a, b \in C_p$, then in case $cw(B) = 0$ we obtain a product

$$\prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} \in B'. \tag{3}$$

Note that if we rearrange elements in (2) as $h_1 h_1^{-1} g_1 g_2^{-1} h_2 h_2^{-1} g_1 g_2^{-1} ... h_p h_p^{-1} g_p g_p^{-1}$ then by the reason of such permutations we obtain a product of corespondent commutators. Therefore, following equality holds true

$$\prod_{i=1}^{p} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} = \prod_{i=1}^{p} h_i g_i h_i^{-1} g_i^{-1} x_0 = \prod_{i=1}^{p} h_i h_i^{-1} g_i g_i^{-1} x \in B', \tag{4}$$

where x_0, x are a products of corespondent commutators. Therefore,

$$(r_1, \ldots, r_{p-1}, r_p) \in W' \text{ iff } r_{p-1} \cdot \ldots \cdot r_1 \cdot r_p = x \in B'. \tag{5}$$

Thus, one element from states of wreath recursion $(r_1, \ldots, r_{p-1}, r_p)$ depends on rest of r_i. This dependence contribute that the product $\prod\limits_{j=1}^{p} r_j$ for an arbitrary sequence $\{r_j\}_{j=1}^{p}$ belongs to B'. Thus, r_p can be expressed as:

$$r_p = r_1^{-1} \cdot \ldots \cdot r_{p-1}^{-1} x.$$

Denote a j-th tuple, which consists of a wreath recursion elements, by $(r_{j_1}, r_{j_2}, ..., r_{j_p})$. Closedness by multiplication of the set of form
$(r_1, \ldots, r_{p-1}, r_p) \in W = (B \wr C_p)'$ follows from

$$\prod_{j=1}^{k} (r_{j1} \ldots r_{jp-1} r_{jp}) = \prod_{j=1}^{k} \prod_{i=1}^{p} r_{j_i} = R_1 R_2 ... R_k \in B', \tag{6}$$

where r_{ji} is i-th element from the tuple number j, $R_j = \prod\limits_{i=1}^{p} r_{ji}$, $1 \leqslant j \leqslant k$. As it was shown above $R_j = \prod\limits_{i=1}^{p-1} r_{ji} \in B'$. Therefore, the product (6) of R_j, $j \in \{1, ..., k\}$ which is similar to the product mentioned in [12], has the property $R_1 R_2 ... R_k \in B'$ too, because of B' is subgroup. Thus, we get a product of form (2) and the similar reasoning as above are applicable.

Let us prove the sufficiency condition. If the set K of elements satisfying the condition of this theorem, that all products of all r_i, where every i occurs in this forms once, belong to B', then using the elements of form

$(r_1, e, ..., e, r_1^{-1})$, ... , $(e, e, ..., e, r_i, e, r_i^{-1})$, ... , $(e, e, ..., e, r_{p-1}, r_{p-1}^{-1})$,
$(e, e, ..., e, r_1 r_2 \cdot ... \cdot r_{p-1})$

we can express any element of form $(r_1, \ldots, r_{p-1}, r_p) \in W = (B \wr C_p)'$. We need to prove that in such way we can express all element from W and only elements of W. The fact that all elements can be generated by elements of K follows from randomness of choice every r_i, $i < p$ and the fact that equality (2) holds so construction of r_p is determined.

Lemma 2. *For any group* B *and integer* $p \geqslant 2$ *if* $w \in (B \wr C_p)'$ *then* w *can be represented as the following wreath recursion*

$$w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j]),$$

where $r_1, \ldots, r_{p-1}, f_j, g_j \in B$ *and* $k \leqslant cw(B)$.

Proof. According to Lemma 1. we have the following wreath recursion

$$w = (r_1, r_2, \ldots, r_{p-1}, r_p),$$

where $r_i \in B$ and $r_{p-1} r_{p-2} \ldots r_2 r_1 r_p = x \in B'$. Therefore we can write $r_p = r_1^{-1} \ldots r_{p-1}^{-1} x$. We also can rewrite element $x \in B'$ as product of commutators $x = \prod_{j=1}^{k} [f_j, g_j]$ where $k \leqslant cw(B)$.

Let us find upper bound of generators number for G'. Let $\mathcal{A}$ be a group and $\mathcal{B}$ a permutation group, i.e. a group $\mathcal{A}$ acting upon a set X, where the active group $\mathcal{A}$ can act not faithfully. Consider the set of all pairs $\{(a, f), f : X \to h, a \in \mathcal{A}\}$. We define a product on this set as

$$\{(a_1, f_1)(a_2, f_2) := (a_1 a_2, f_1 f_2^{a_1})\},$$

where $f_1^{a_2}(x) = f_1(a_2(x))$.

Theorem 3. *If* $W = (\mathcal{A}, X) \wr (\mathcal{B}, Y)$, *where* $|X| = n$, $|Y| = m$ *and active group* $\mathcal{A}$ *acts on* X *transitively, then*

$$d(G') \leqslant (n-1) d(\mathcal{B}) + d(\mathcal{B}') + d(\mathcal{A}').$$

Proof. The generators of W' in form of tableaux [2]:

$a'_i = (a_i; e, e, e, \ldots, e)$, $t_1 = (e; h_{j_1}, e, e, \ldots, c_{j_1}), \ldots,$ $t_k = (e; e, e, e, \ldots, h_{j_k}, e, \ldots, c_{j_k})$, $t_l = (e; e, e, e, \ldots$

where $h_j, c_{j_l} \in S_B$, $\mathcal{B} = \langle S_B \rangle$, $a_i \in S_A$, $A = \langle S_A \rangle$. Note that, on a each coordinate of tableau, that presents a commutator of $[a; h_1, \ldots, h_n]$ and $[b; g_1, \ldots, g_n]$, $a, b \in \mathcal{A}, h_i, g_j \in \mathcal{B}$ can be product of form $a_1 a_2 a_1^{-1} a_2^{-1} \in \mathcal{A}'$ and $h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} \in \mathcal{B}$, according to Corollary 4.9 [12]. This products should satisfy the following condition:

$$\prod_{i \in X}^{n} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} \in \mathcal{B}'. \tag{7}$$

That is to say that the product of coordinates of wreath product base is an element of commutator $\mathcal{B}'$. As it was described above it is subdirect product of $\underbrace{\mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}}_{n}$ with the additional condition (8). This is the case because not all element of the subdirect product are independent because the elements must be chosen in such a way that (8) holds. We may rearrange the factors in the product in the following way:

$$\prod_{i=1}^{n} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} = (\prod_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1})[g, h] \in \mathcal{B}'.$$

where $[g,h]$ is a commutator in case $cw(B) = 1$. We express this element from $\mathcal{B}'$ as commutator $[g,h]$ if $cw(B) = 1$. In the general case, we would have $\prod\limits_{j=1}^{cw(B)} [g_j, h_j]$ instead of this element. This commutator are formed as product of commutators of rearranged elements of $\prod\limits_{i=1}^{n} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1}$. Therefore, we have a subdirect product of n the copies of the group B which has been equipped by condition (8). The multiplier $\prod\limits_{j=1}^{cw(B)} [g_j, h_j]$ from $\mathcal{B}'$, which has at least $d(\mathcal{B}')$ generators

$$\prod_{i=1}^{n} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} = (\prod_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1}) \prod_{j=1}^{cw(B)} [g_j, h_j] \in \mathcal{B}'.$$

Since $(\prod\limits_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1}) = e$ and the product $\prod\limits_{j=1}^{cw(B)} [g_j, h_j]$ belongs to $\mathcal{B}'$, then condition (8) holds. The assertion of a theorem on a recursive principle is easily generalized on multiple wreath product of groups.

Thus minimal total amount consists of at least $d(\mathcal{B}')$ generators for $n-1$ factors of group $\mathcal{B}$, $d(\mathcal{B}')$ generators for the dependent factor from $\mathcal{B}'$ and $d(\mathcal{A})$ generators of the group $\mathcal{A}$ which concludes the proof.

It should be noted that not all the elements of commutator subgroup, that has structure of the subdirect product, are independent by (8), at least one of them must be chosen carefully such that would be (8) satisfied. This implies the estimation $d(G') \leqslant (n-1)d(B) + d(B')$.

Thus minimal total amount consists of at least $d(\mathcal{B}')$ generators for $n-1$ factors of group $\mathcal{B}$, $d(\mathcal{B}')$ generators for the dependent factor from $\mathcal{B}'$ and $d(\mathcal{A})$ generators of the group $\mathcal{A}$ which concludes the proof.

Let us find upper bound of generators number for G'. Let $\mathcal{A}$ be a group and $\mathcal{B}$ a permutation group, i.e. a group $\mathcal{A}$ acting upon a set X, where the active group $\mathcal{A}$ can act not faithfully. Consider the set of all pairs $\{(a,f), f : X \to h, a \in \mathcal{A}\}$. We define a product on this set as

$$\{(a_1, f_1)(a_2, f_2) := (a_1 a_2, f_1 f_2^{a_1})\},$$

where $f_1^{a_2}(x) = f_1(a_2(x))$.

Theorem 4. *If $W = (\mathcal{A}, X) \wr (\mathcal{B}, Y)$, where $|X| = n$, $|Y| = m$ and active group $\mathcal{A}$ acts on X transitively, then*

$$d(G') \leqslant (n-1)d(\mathcal{B}) + d(\mathcal{B}') + d(\mathcal{A}').$$

Proof. The generators of W' in form of tableaux [2]: $a'_i = (a_i; e, e, e, \ldots, e)$,

$$t_1 = (e; h_{j_1}, e, e, \ldots, c_{j_1}), \ldots, t_k = (e; e, e, e, \ldots, h_{j_k}, e, \ldots, c_{j_k}), \quad t_l = (e; e, e, e, \ldots, h_{j_l}, c_{j_l}),$$

where $h_j, c_{j_l} \in S_B$, $\mathcal{B} = \langle S_B \rangle$, $a_i \in S_A$, $A = \langle S_A \rangle$. Note that, on a each coordinate of tableau, that presents a commutator of $[a; h_1, \ldots, h_n]$ and $[b; g_1, \ldots, g_n]$, $a, b \in \mathcal{A}, h_i, g_j \in \mathcal{B}$ can be product of form $a_1 a_2 a_1^{-1} a_2^{-1} \in \mathcal{A}'$ and $h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} \in \mathcal{B}$, according to Corollary 4.9 [12]. This products should satisfy the following condition:

$$\prod_{i \in X}^{n} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} \in \mathcal{B}'. \tag{8}$$

That is to say that the product of coordinates of wreath product base is an element of commutator $\mathcal{B}'$. As it was described above it is subdirect product of $\underbrace{\mathcal{B}\times\mathcal{B}\times\cdots\times\mathcal{B}}_{n}$ with the additional condition (8). This is the case because not all element of the subdirect product are independent because the elements must be chosen in such a way that (8) holds. We may rearrange the factors in the product in the following way:

$$\prod_{i=1}^{n} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} = (\prod_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1})[g,h] \in \mathcal{B}'.$$

where $[g,h]$ is a commutator in case $cw(B) = 1$. We express this element from $\mathcal{B}'$ as commutator $[g,h]$ if $cw(B) = 1$. In the general case, we would have $\prod\limits_{j=1}^{cw(B)} [g_j,h_j]$ instead of this element. This commutator are formed as product of commutators of rearranged elements of $\prod\limits_{i=1}^{n} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1}$. Therefore, we have a subdirect product of n the copies of the group B which has been equipped by condition (8). The multiplier $\prod\limits_{j=1}^{cw(B)} [g_j,h_j]$ from $\mathcal{B}'$, which has at least $d(\mathcal{B}')$ generators

$$\prod_{i=1}^{n} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} = (\prod_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1}) \prod_{j=1}^{cw(B)} [g_j,h_j] \in \mathcal{B}'.$$

Since $(\prod\limits_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1}) = e$ and the product $\prod\limits_{j=1}^{cw(B)} [g_j,h_j]$ belongs to $\mathcal{B}'$, then condition (8) holds. The assertion of a theorem on a recursive principle is easily generalized on multiple wreath product of groups.

Thus minimal total amount consists of at least $d(\mathcal{B}')$ generators for $n-1$ factors of group $\mathcal{B}$, $d(\mathcal{B}')$ generators for the dependent factor from $\mathcal{B}'$ and $d(\mathcal{A})$ generators of the group $\mathcal{A}$ which concludes the proof.

It should be noted that not all the elements of commutator subgroup, that has structure of the subdirect product, are independent by (8), at least one of them must be chosen carefully such that would be (8) satisfied. This implies the estimation $d(G') \leqslant (n-1)d(B) + d(B')$.

Thus minimal total amount consists of at least $d(\mathcal{B}')$ generators for $n-1$ factors of group $\mathcal{B}$, $d(\mathcal{B}')$ generators for the dependent factor from $\mathcal{B}'$ and $d(\mathcal{A})$ generators of the group $\mathcal{A}$ which concludes the proof. □

We shall consider special case when a passive group $(\mathcal{B}, Y)$ of W is a perfect group. Since we obtain a direct product of $n-1$ the copies of the group B then according to Corollary 3.2. of Wiegold J. [24] $d(\mathcal{B}^n) \leqslant d(\mathcal{B}) + n - 1$ [24]. More exact upper bound give us Theorem A. [24], which use s a the size of the smallest simple image of G.

Therefore, in this case our upper bound has the form

$$d(W') \leqslant clog_s n + d(\mathcal{B}') + d(\mathcal{A}').$$

Now we consider no regular wreath product, where active group can be both as infinite as finite and consider a center of such group. This is generalization of Theorem 4.2 from the book [12] because action of $\mathcal{A}$ is not non faithfully. Let $X = \{x_1, x_2, \ldots, x_n\}$ be $\mathcal{A}$-space. If an non

faithfully action by conjugation determines a shift of copies of $\mathcal{B}$ from direct product $\mathcal{B}^n$ then we have not standard wreath product $(\mathcal{A}, X) \wr \mathcal{B}$ that is semidirect product of $\mathcal{A}$ and $\prod_{x_i \in X} \mathbb{B}$ that is $\mathcal{A} \ltimes_{\varphi} (\mathcal{B})^n$ and the following Proposition holds. Let $\mathcal{K} = ker(\mathcal{A}, X)$ that is subgroup of $\mathcal{A}$ that acts on X as a pointwise stabiliser, that is kernel of action of $\mathcal{A}$ on X.

Denote by $Z(\tilde{\triangle}(\mathcal{B}))$ the subgroup of diagonal subgroup [4] $Fun(X, Z(B))$ of functions $f : X \to Z(B)$ which are constant on each orbit of action of A on X for unrestricted wreath product, and denote by $Z(\triangle(\mathcal{B}^n))$ the subgroup of diagonal $Fun(X, Z(\mathcal{B}^n))$ of functions with the same property for restricted wreath product, where n is number of non-trivial coordinates in base of wreath product.

Proposition 1 A center of the group $(\mathcal{A}, X) \wr \mathcal{B}$ is direct product of normal closure of center of diagonal of $Z(\mathcal{B}^n)$ i.e. $(E \times Z(\triangle(\mathcal{B}^n)))$, trivial an element, and intersection of $(\mathcal{K}) \times E$ with $Z(\mathcal{A})$. In other words,

$$Z((\mathcal{A}, X) \wr \mathcal{B}) = \langle (1;\ \underbrace{h, h, \ldots, h}_{n}),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq (Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\triangle(\mathcal{B}^n)),$$

where $h \in Z(\mathcal{B})$, $|X| = n$.

For restricted wreath product with n non-trivial coordinates: $Z((\mathcal{A}, X) \wr \mathcal{B}) =$
$\langle (1;\ \ldots, h, h, \ldots, h, \ldots),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq (Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\triangle(\mathcal{B}^n))$.

In case of unrestricted wreath product we have: $Z((\mathcal{A}, X) \wr \mathcal{B}) =$
$\langle (1;\ \ldots, h_{-1}, h_0, h_1, \ldots, h_i, h_{i+1}, \ldots,),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq (Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\tilde{\triangle}(\mathcal{B}))$.

Proof. The elements of center subgroup have to satisfy the condition: $f : X \to B$ such is constant on each orbit $\mathcal{O}_j$ of action $\mathcal{A}$ on X i.e. $f(x) = b_i$ for any $x \in \mathcal{O}_j$. Also every $b_x :\ b_x \in Z(\mathcal{B})$. Indeed the elements of form $(1;\ \underbrace{h, h, \ldots, h}_{n})$ will not be changed by action of conjugation of any element from $\mathcal{A}$ because any permutation elements coordinate of diagonal of $\mathcal{B}^n$ does not change it. Also h commutes with any element of base of $(\mathcal{A}, X) \wr \mathcal{B}$ because h from center of $\mathcal{B}$. Since the action is defined by shift on finite set X, $|X| = n$ is not faithfully, then its kernel $\mathcal{K} \neq E$ which confirms the proposition. Also elements of subgroup $(\mathcal{A}, X) \wr \mathcal{E})$ belongs to $Z((\mathcal{A}, X) \wr \mathcal{B})$ iff it acts trivial on X. □

Example 1 If $\mathcal{A} = \mathbb{Z}$ then a center $Z((\mathbb{Z}, X) \wr \mathcal{B}) =$
$\langle (1;\ \underbrace{h, h, \ldots, h}_{n}),\ e,\ n\mathbb{Z} \ltimes \mathcal{E} : h \in Z(\triangle(\mathcal{B}^n)) \rangle$. Since the action defined by shift on finite set X is not faithfully, and its kernel is isomorphic to $n\mathbb{Z}$ because cyclic shift on n coordinates is invariant on X.

Generating set for commutator subgroup $(\mathbb{Z}_n \wr \mathbb{Z}_m)'$, where $\mathbb{Z}_n$, $\mathbb{Z}_m$ have presentation in additive form, is the following:

$$\begin{aligned} h_1 &= (0; 1, 0, \ldots, m-1), \\ h_2 &= (0; 0, 1, 0, \ldots, m-1), \\ &\vdots \\ h_{n-1} &= (0; 0, \ldots, 1, m-1). \end{aligned}$$

Thus, it consist of n tableaux of form $h_i = (h_{i1}, \ldots, h_{im})$ and relations for coordinate of any tableau $h_i, i \in \{1, \ldots, n-1\}$ is

$$h_{i1} + \cdots + h_{in-1} \equiv 0 (\text{mod } m).$$

According to Theorem 3, for wreath product of abelian groups presented in multiplicative form, this relation has the form

$$\prod_{i=1}^{n} h_i f_{i_{\pi_a}} h^{-1}_{i_{\pi_a\pi_b}} f^{-1}_{i_{\pi_a\pi_b\pi_a^{-1}}} [h,f] = \prod_{i=1}^{n} (h_i f_{i_{\pi_a}} h^{-1}_{i_{\pi_a\pi_b}} f^{-1}_{i_{\pi_a\pi_b\pi_a^{-1}}} \prod_{j=i+1}^{i+2} [h_j, f_{j_{\pi_a}}]) = e.$$

Example 2 If $G = \mathbb{Z}_n \wr \mathbb{Z}_m$ is standard wreath product, then $d(G') = n-1$.

Let $G = Z \wr_X Z$ and $G = A \wr_X B$ be a restricted wreath product, where only n non-trivial elements in coordinates of base of wreath product which are indexed by elements from X, in degenerated case $\mid X \mid = n$. Z acts on X by left shift. Also A acts transitively from left.

Remark 1 The quotient group of a restricted wreath products $G = Z \wr_X Z$ by a commutator subgroup is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. In previous conditions if $G = A \wr_X B$ then, $G/G' = A/A' \times B/B'$. If $G = Z_n \wr Z_m$, where $(m,\ n) = 1$, then $d(G/G') = 1$. If $G = Z \wr Z$ is an unrestricted regular wreath product then $G/G' \simeq Z \times E \simeq Z$.

Proof. Consider the element of $G = A \wr_X \mathcal{B}$, where A can be Z which acts on X by left shift, then elements of commutator subgroup has form:
$[e; \ldots, h_{-n}, \ldots, h_0, h_1, \ldots, h_n, \ldots,]$, where $h_i \in B$. According to Corollary 4.9 [12] the commutator of elements $h = [a; h_1, \ldots, h_n]$, $g = [b; g_1, \ldots, g_n]$, $g, h \in G$ satisfies the condition (8), which for case where B is abelian such: $\prod\limits_{i=1}^{n} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} = e$, where g_i, h_i are non trivial coordinates from base of group, $a,\ b \in A$, $g_i, h_j \in B$. The commutator with the shifted coordinate $h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)}$ appears within the i-th coordinate position due to action of A. According to Corollary 4.9 [12] the set of elements satisfying condition (8) forms a commutator. Also the equivalent condition can be formulated:

$$\prod_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1} \in \mathcal{B}', \tag{9}$$

Therefore, if $\mathcal{B}$ is abelian an element h of G belongs to G' iff h satisfy a condition: $\prod\limits_{i=1}^{n} h_i = e$.

For unrestricted wreath product to show that all base of wreath product is in the commutator subgroup we choose an element $[e; \ldots, h_{-1}, h_0, h_1, \ldots]$, where h_i is variable, and form a commutator which is an arbitrary element $[e; \ldots, g_{-1}, g_0, g_1, \ldots]$ of wreath product base:
$[e; \ldots, h_{-1}, h_0, h_1, \ldots][\sigma; e, e, \ldots, e][e; \ldots, h_{-1}^{-1}, h_0^{-1}, h_1^{-1}, \ldots][\sigma^{-1}; e, e, \ldots, e] =$
$= [e; \ldots, g_{-1}, g_0, g_1, \ldots]$. For convenience we present Z in additive form. Then to previous equality holds the following equations have to be satisfied: $h_0 - h_1 = g_0, h_1 - h_2 = 0, h_2 - h_3 = 0, \ldots$. it implies that $h_1 = h_0 - 1$, $h_2 = h_1$, $h_3 = h_2$, ... $h_i + 1 = h_i$. Therefore $h_i = 0$, $i \geqslant 1$. From other side we have $h_{-1} - h_0 = g_0, h_{-1} - h_{-2} = 0, h_{-2} - h_{-3} = 0, \ldots$. so $h_{-i} = g_0$, for all $i < 0$. That is impossible in the restricted case but possible in the unrestricted. As a corollary $G/G' \simeq Z \times Z$ for restricted case. Thus, for unrestricted case all base of G is in G' as a corollary $G/G' \simeq Z \times E$.

Thus, this group is a subdirect product of $\underbrace{B \times B \times \cdots \times B}_{n}$ with the additional condition (9) where, because for any element of the subgroup of coordinates there exists a surjective homomorphism acting upon B, we can conclude that G' must be a subdirect product. The commutator subgroup is the kernel of homomorphism $\varphi : G \twoheadrightarrow G/G'$. More precisely,

$G = (Z, X) \wr (Z, Y) \twoheadrightarrow G/G'$
$\simeq Z/Z' \times Z/Z' = \mathbb{Z} \times \mathbb{Z}$. In case $G = A \wr \mathcal{B}$ the $ker\varphi$ has the same structure, the homomorphism φ maps those elements of B^n, as base of G, which satisfy $\prod_{i=1}^{n} h_i = e$, i.e. the elements of B' in e of the group G/G'. Thus, $ker\varphi = G'$. To show that the properties of injectivity and surjectivity hold for this homomorphism, we chose the elements from G which have the form $[e; e, \ldots e, h, e, \ldots, e]$, where $h \notin G'$, corresponding to a specimen from the quotient group B/B'. Also we chose independently, an element of the form $[a; e, \ldots, e, \ldots, e]$ corresponding to a specimen of the quotient group A/A'. Therefore, we must have a one-to-one correspondence between G/G' and $A/A' \times B/B'$. In this case, we obtain ${}^{G}/_{G'} \simeq \left[{}^{A}/_{A'} \times {}^{B}/_{B'}\right]$.

In the scenario when the action of Z upon the n elements from the set is isomorphic to the action of Z_n elements on the set or the action of the Z_n elements on itself. In case $G = Z \wr Z$ we have ${}^{G}/_{G'} \simeq [\mathbb{Z} \times \mathbb{Z}]$. □

For the group $G = Z_n \wr Z_m$ the same is true with ${}^{G}/_{G'} \simeq [\mathbb{Z}_n \times \mathbb{Z}_m]$ and dependently of fact of $(m, n) = 1$ or not can admits one or two generators. For the group $G = Z_n \wr Z_m$ it should be noted that the same is true. In the general case, $\wr_{i=1}^{n} \mathbb{Z}_{m_i}$ can have only one generator more than the quotient by commutator has.

Application to Geometric Groups of Diffeomorphisms Acting on the Möbius Band

Maksymenko S. [11] studied various different geometric objects and considered the actions of diffeomorphisms on them. We now consider the algebraic structure and the generators for a group of such type.

Let M be a smooth compact connected surface, P be either the real line or the circle, $f : M \to P$ be a smooth map, and $O(f)$ be the orbit of f with respect to the right action of the group $\mathrm{Diff}(M)(\mathcal{D}(M))$ of diffeomorphisms of M. We assume that at each critical point, the map f is equivalent to a homogeneous polynomial in two variables without multiple factors. Conversely, it should be noted that every group obtained in the way described will be isomorphic to $G(f)$ for some smooth map $f : M \to P$.

We will now specify the object and the construction of orbits under the action of the group diffeomorphisms. Let $f : M \to \mathbb{R}$ be a Morse function on a connected compact surface M. Let $\mathcal{S}(f)$ and $O_f = \mathcal{O}(f)$ be the stabiliser and the orbit of f with respect to the right action of the group of diffeomorphisms $\mathcal{D}(M)$ respectively.

Let X_f denote the partition of M whose elements are the connected components of level-sets $f^{-1}(c)$ of f. It should be noted here that an element $g \in X_f$ is called critical if it contains a critical point of f, otherwise, the elements is called regular. It is well known within this research domain that the factor space M/X_f, has a natural structure of a finite graph and is entitled the Kronrod-Reeb graph.

In our case, the diffeomorphisms act upon the Möbius band. Let M now be a compact not orientable surface and ω be a volume from M which has h-form of the Möbius band. For a smooth map $f : M \to R$, denote by $S(f)$, the subgroup $\mathcal{D}(M)$ of diffeomorphisms h (of M) which preserve f, i.e. those satisfying the relation $f \circ h = f$.

This group is associated with $S_{id}(f)$, which is a subgroup of stabiliser elements isotopic to the identity, i.e. $\pi_1(O_f, f) \simeq \pi_0 S_{id}(f)$, where the last isomorphism arises due to locally trivial bundle. Because there are a locally trivial bundle of homotopical groups with base $\pi_1(O_f,\ f)$ and layer $\pi_0 S_{id}(f)$, this means that an exact sequence of homotopic groups and locally trivial bundle of homotopical groups give an explanation of the isomorphism $\pi_1(O_f, f) \simeq \pi_0 S_{id}(f)$.

This locally trivial bundle of homotopical groups induce an exact sequence of homotopic groups of that bundle. Now since the group of diffeomorphisms is infinitely dimensional, we have found the connected components. This group is associated with the action of the group $\frac{S(f)}{S_{id}(f)}$ upon splitting into the function level lines f.

In our case, the Morse function upon M has two local extremes, which are the points of local maximum. Moreover, the Morse function f must have critical sets with exactly one saddle point. The lines of levels around a local maximum point of f have the form of coaxial circles, where these lines are determined by the polynomial with form $\pm(x^2+y^2)+c$.

Being a little more precise, we will now consider the function of Morse f on M, which satisfies the following three properties:

1. f is constant on the bound M;
2. there are two points of maximum at a saddle point;
3. at the two points of maximum, the values of the function are equal, i.e. at every critical point of f, the germ of f is C^∞ equivalent to some homogeneous polynomial in two real variables without multiple factors.

Let $f : M \to R$ now be a C^∞ Morse function. We note here that since the polynomial $\pm(x^2+y^2)+c$ is homogeneous and has no multiple factors, it follows (from the celebrated Morse Lemma) that the space of all Morse maps belongs to the space of maps $\mathcal{F}(\mathcal{M},\mathcal{P})$, where f here only has isolated critical points and $\mathcal{P}$ is either the real line $\mathbb{R}$ or the circle S^1.

Let $\mathcal{D}(M)$ be a group of diffeomorphisms which preserve the Morse function f on M. We know from the results of Maksymenko S. [11] that $\pi_0(D(M)) \simeq \mathbb{Z}$. Let there exist upon M, n identical regions X_i (critical sets) which have, for example, the form of doubles, meaning that f has two critical points in each X_i and additionally, that X_i are the domains of simple connectedness.

Consider a group H of automorphisms of M which are induced by the action of diffeomorphisms h of a group $D(M)$ which preserve the Möbius function f. In other words, the h here are from the stabiliser $S(f) \lhd D(M)$. We note that the generators with stabilisers with the right action by diffeomorphisms $\pi_0 S(f|_{X_i,\partial X_i})$ are τ_i. The generators of the cyclic group Z which define a shift are ρ. Since the group action is continuous, this implies that the ρ can realize only cyclic shifts, else one would change the domains of simplicity X_i order.

Assume there are n critical sets X_i on M. The automorphism group $H \simeq \pi_1(O_f, f)$ has exactly two subgroups $\mathbb{Z}$ which correspond to the rotation of M, whose critical sets X_i have not changed the order of X_i and $(\mathbb{Z})^n$ denotes the subgroup of automorphisms of n critical sets. Analogously to previous investigations [11, 16, 18], there exists a short exact sequence $0 \to \mathbb{Z}^m \to \pi_1(O_f, f) \to \mathbb{Z} \to 0$, where the G-group of automorphisms are Reeb's (Kronrod-Reeb) graph [11] and hence $O_f(f)$ is an orbit under action of diffeomorphism group.

The application of such an action results in a surjective epimorphism to a group $\mathbb{Z}$, which has the left inverse and arises as a result of splitting. The automorphism group therefore has the structure of a semi-direct product $(\mathbb{Z})^n \rtimes \mathbb{Z}$. This is in agreement with the work of Maksymenko S. [11] who considers a similar scenario but for a different group and set (surface). Moreover, we note that this Morse function f has critical sets X_i on Möbius band (M) with one saddle point.

The minimal set of generators for the fundamental group $\pi_1(O_f, f)$ of the orbit of the function f with respect to the action of the group of diffeomorphisms of non-moving ∂M is found in the next theorem.

We note that since the action of the group of diffeomorphisms on n-critical sets of M have been determined and described, the next thing to be considered is how this group has the correspondent to this action structure $\pi_1(O_f, f) \simeq \mathbb{Z} \ltimes (\mathbb{Z})^n$. We will denote by H, the fundamental group $\pi_1(O_f, f)$.

The first generator is ρ since it realises the shift of the Möbius band. The second, τ, realises the rotation of domains X_i of simple connectedness upon the Möbius band when passing through its twisting point. In other words, τ acts by the automorphism by permutation of sheets of doubles X_i with the winding of outer adjacency on each double X_i. Thus, we conclude that τ_i has infinite order.

Bounds on these domains are the lines of levels of function f upon these domains, or in other words, the sets of points with $f = const_1$. We shall now prove that the action of the first generator ρ of the group defines the homomorphism in $Aut(Z^n)$..

Theorem 5. *The group $H \simeq \mathbb{Z} \ltimes (\mathbb{Z})^n = \langle \rho,\ \tau \rangle$ with defined above homomorphism in $AutZ^n$ has two generators and non trivial relations*

$$\rho^n \tau \rho^{-n} = \tau^{-1},\ \rho^i \tau \rho^{-i} \rho^j \tau \rho^{-j} = \rho^j \tau \rho^{-j} \rho^i \tau \rho^{-i},\ 0 < i, j < n.$$

Also this group admits another presentation in generators and relations

$$\left\langle \rho, \tau_1, \ldots, \tau_n \,\middle|\, \rho \tau_{i(\ mod\ n)} \rho^{-1} = \tau_{i+1(\ mod\ n)}\ ,\ \ \tau_i \tau_j = \tau_j \tau_i,\ i,\ j \leqslant n \right\rangle. \tag{10}$$

Proof. From the description above, we have that the action of the first generator of the group, ρ, defines the homomorphism in the $Aut(Z^n)$. There exists such a diffeomorphism from $D(M)$, called the Dehn twist, which has an infinite order since it makes a winding of outer adjacency (refer to Dehn twist in [6]) on the doubles X_i, and it belongs to stabiliser $S(f)$. The generator τ must therefore correspond to this diffeomorphism.

Let x_i denote the number of domains from X_i. The action of the first generator ρ, defines the homomorphism $\rho(x_1, \ldots, x_n) = \varphi^\rho(x_1, \ldots, x_n)$, where $\varphi(x_1, \ldots, x_n) = (-x_n, x_1, \ldots, x_{n-1})$. It should be noted that this action could be equivalently represented as

$$\varphi^\rho(x_1, \ldots, x_n) = ((-1)^{\left[\frac{\rho+n-1}{n}\right]}(-x_{(1-\rho)modn}), \ldots,$$

$$(-1)^{\left[\frac{\rho+n-k}{n}\right]} x_{(k-\rho)modn}, \ldots, (-1)^{\left[\frac{\rho}{n}\right]} x_{(n-\rho)modn} x_n).$$

We extend the action of ρ onto an arbitrary $\alpha \in \mathbb{Z}$. This action involves sequential shifts of X_i along the orbit on M defined as α. We have

$$\varphi^\alpha(x_1, \ldots, x_n) = ((-1)^{\left[\frac{\alpha+n-1}{n}\right]}(-x_{(1-\alpha)modn}), \ldots$$

$$\ldots, (-1)^{\left[\frac{\alpha+n-k}{n}\right]} x_{(k-\alpha)modn},\ \ldots\ , (-1)^{\left[\frac{\alpha}{n}\right]} x_{(n-\alpha)modn} x_n).$$

We will now consider two sets. The first set is $\alpha = \{1, 2, \ldots, n\}$. As an example, if $\alpha = 1$, then we have $\left[\frac{\alpha+n-1}{n}\right] = \left[\frac{1+n-1}{n}\right] = 1$. Additionally, we find the numbers $m \in N$ such that $\left[\frac{m+n-1}{n}\right] = 1$ and the numbers that are congruent to these m modulo $2n$.

The second set is $\alpha = \{0, -1, -2, \ldots, -n+1\}$. Similarly, we note that congruence modulo $2n$ is of interest. As an example, if $\alpha = -1$, then we have $\left[\frac{\alpha+n-1}{n}\right] = \left[\frac{-1+n-1}{n}\right] = 0$. Hence, $\alpha \in \mathbb{Z}$, $\alpha = l\rho$ is the number of shifts defined by α, while τ_i corresponds to the action of automorphism by permutation with winding of outer adjacency (Dehn twist in [6]) on the

doubles X_i. We conclude that τ_i therefore has an infinite order due to the Dehn twist. The value of the sign of the x_i indicates the presence of a rotation of the doubles or its absence.

The relations for the non-minimal generating set is precisely

$$\left\langle \rho, \tau_1, \ldots, \tau_n \left| \rho\tau_{i(\mathrm{mod}n)}\rho^{-1} = \tau_{i+1(\mathrm{mod}n)} \right. \right\rangle.$$

This formulation yields that the relations for the minimal generating set is $\langle \rho,\, \tau \rangle$ are

$$\left\langle \rho^{2n}\tau_1\rho^{-2n} = \tau_1,\ |\tau_1| = |\rho| = \infty, \tau_1 = \tau \right\rangle,$$

where $\tau_1 = \tau$ and since $\rho^{2n}\tau_i\rho^{-2n} = \tau_{i+1}$ we transform our minimal generating set into a canonical generating set of the $n+1$ elements given by $\langle \rho,\, \tau_1, \tau_2, \ldots, \tau_n \rangle$.

It is known that the generators of the semidirect product $G \ltimes H$ may be presented in the form (g, h). We now utilise this form to say that the generators of Z^n have the form of vectors $\tau_1 = (h_1, e, e, \mathrm{B}\ldots, e,)$, $\tau_2 = (e, h_2, e, \mathrm{B}\ldots, e,), \mathrm{B}\ldots$, $\tau_n = (e, e, \mathrm{B}\ldots, h_n)$. Making us of the operation of conjugation for $(e, h_1) = \tau_1 = \tau$, allows us to express the second generator of Z^n. Note that this is by $(g, e) = \rho$, where h_1 is one of generators of Z^n and g is generator of Z.

$$(g,e)^{-1}\ (e,\ \tau_1)\,(g,e) =\ (e,\ \tau_2).$$

Analogously, we find

$$(g,e)^{-1}\ (e,\ \tau_2)\,(g,e) =\ (e,\ \tau_3),$$

and, for a general term, we find

$$(g,e)^{-1}\ (e,\ \tau_{n-1})\,(g,e) =\ (e,\ \tau_n).$$

We show that there are not otherwise independent relations within the group H. For this group H, all canonical words have the form

$$\rho^k\tau_1^{s_1}\tau_2^{s_2}\ldots\tau_n^{s_n}. \tag{11}$$

This form follows from the form of semidirect product elements. We now prove using (10) and reductions of reciprocals elements, that we may transform any finite non-trivial word of F_{n+1} to the form (11). Additionally, we shall prove that the set of all words which maps trivially by a surjective homomorphism, with a kernel which is a normal closure of relations from the set R from (10), are those coincides with trivial words in the group $\pi(O_f, f)$. For this purpose, we prove the transformation equivalence $\tau_i\rho = \rho\tau_{i+1}$. In fact,

$$\tau_{i(\mathrm{mod}n)}\rho = \rho\rho^{-1}\tau_{i(\mathrm{mod}n)}\rho = \rho\tau_{(i+1)(\mathrm{mod}n)}. \tag{12}$$

It should be noted that the relation $\tau_i\tau_j = \tau_j\tau_i$ holds since automorphisms of X_i and X_j are independent of each other. Therefore, using this transformation, we can rearrange all the ρ to be in the first position in the word over the alphabet $\{\rho, \tau_1, \ldots, \tau_n\}$.

We will show that normal closure of the relations $\rho\tau_{i(\mathrm{mod}n)}\rho^{-1} = \tau_{i+1(\mathrm{mod}\ n)}$, with $\tau_i\tau_j = \tau_j\tau_i$, determines the kernel of the surjective homomorphism ψ from F_{n+1} to H. The images of such a mapping are the canonical words (11) which have the form of H. The form of these canonical words are determined by the semidirect product $\mathbb{Z} \ltimes (\mathbb{Z})^n$ and its automorphisms. This mapping φ has the form

$$x_{j_1}^{p_1}x_{j_2}^{p_2}x_{j_3}^{p_3}\ \ldots\ x_{j_m}^{p_m} \mapsto \rho^k\tau_1^{s_1}\tau_2^{s_2}\ \ldots\ \tau_n^{s_n},$$

where $x_{j_i} \in F_{n+1}$, $x_{j_1}^{p_1} x_{j_2}^{p_2} x_{j_3}^{p_3} \ldots x_{j_m}^{p_m} \in F_{n+1}$ and $\sum\limits_{i=1}^{n} s_i + k = \sum\limits_{l=1}^{m} p_l$.

For this purpose, we use the transformation equivalence $\tau_i \rho = \rho \tau_{i+1}$. Making use this transformation allows us to therefore rearrange all ρ to the first position in the word over the alphabet $\{\rho, \tau_1, \ldots, \tau_n\}$. Being a little more precise, this conversion is expressed as

$$\tau_{i(\mathrm{mod} n)} \rho = \rho \rho^{-1} \tau_{i(\mathrm{mod} n)} \rho = \rho \tau_{(i+1)(\mathrm{mod} n)}.$$

The kernel of surjective homomorphism ψ contains exactly those words that, after mapping, ψ becomes the trivial words in the group H since those trivial words have the form $\rho^0 \tau_1^0 \tau_2^0 \ldots \tau_n^0$.

Note that an arbitrary word from $ker(\psi)$ may be transformed due to (12) into $\rho^0 \tau_1^{i_1} \tau_2^{i_2} \ldots \tau_n^{i_n}$, where $i_k = 0$ for all k. In fact, $ker(\psi)$ is the normal closure of the relations (10) and hence it consist of the words $\rho \tau_{i(\mathrm{\ mod\ } n)} \rho^{-1} \tau_{i+1(\mathrm{\ mod\ } n)}^{-1}$, with $[\tau_i, \tau_{i+1}]$.

In particular, the word $\rho \tau_{i(\mathrm{mod}\ n)} \rho^{-1} \tau_{i+1(\mathrm{mod} n)}^{-1}$ transforms by (12) to $\rho \rho^{-1} \tau_{i+1(\mathrm{mod} n)} \tau_{i+1(\mathrm{mod} n)}^{-1}$. The words from the normal closure must therefore have zero sum of powers for each generator.

In the real group H, with the reduced canonical words (11), where all generators have infinite order, only those words with zero exponents of generators are trivial. We have therefore found all such relations. which concludes the proof. □

It should be noted that the main property of the homomorphism φ, from F_{n+1} onto H, holds due to the same transformation (12). We now consider

$$\begin{aligned} \varphi(ab) = \varphi(a)\varphi(b) &= \varphi\left(x_{j_1}^{p_1} x_{j_2}^{p_2} \ldots x_{j_m}^{p_m}\right) \varphi\left(x_{i_1}^{q_1} x_{i_2}^{q_2} \ldots x_{i_m}^{q_m}\right) \\ &= \rho^k \tau_1^{s_1} \tau_2^{s_2} \ldots \tau_n^{s_n} \rho^m \tau_1^{j_1} \tau_2^{j_2} \ldots \tau_n^{j_n} = \rho^{k+m} \tau_1^{f_1} \tau_2^{f_2} \ldots \tau_n^{f_n} \end{aligned}$$

Thus, the main property of the homomorphism holds. It should be noted that such a relation $\rho^{2n} \tau_1 \rho^{-2n} = \tau_1$ is typical for a wreath product.

The homomorphism from the group Z into the group $Aut\mathrm{Z}^n$, determining a shift of generators $(\tau_1, \ldots, \tau_n)$ of Z^n, can be equivalently presented by the matrix ϕ. For the case $n = 4$, ϕ has the form

$$\phi = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The generators of the subgroup Z^n can be presented in the form of vectors. These vectors are precisely

$$(h_1, e, e, e), (e, h_2, e, e), \ldots, (e, e, e, h_4).$$

In order to check the relation for the case $n = 4$, we consider

$$\phi^4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Thus, $\phi^8 = E$ and our relation $\rho^{2n} \tau_1 \rho^{-2n} = \tau_1$ holds.

It should be noted that the research of Maksymenko S. [11] tells us that a group of this kind arises as a fundamental group of the orbit $\pi_1(O_f, f)$ for some Morse function f which, as described above, acts upon the Möbius band M.

Note that we have derived the relation $\rho^{2n}\tau\rho^{-2n} = \tau$. If we now multiply this from left on τ^{-1}, we can equivalently express this as

$$\tau^{-1}\rho^{2n}\tau\rho^{-2n} = e.$$

In a similar fashion, the multiplication from the right on ρ^{2n} obtains

$$\tau^{-1}\rho^{2n}\tau = \rho^{2n}.$$

One such relation characterises the Bauslag-Soliter group. This is the group $G(m;k)$, which has the form $G(m;n) = \langle a, b; a^{-1}b^m a = b^k \rangle$, where $m, k \in \mathbb{Z}$. Note the Bauslag-Soliter group has only one relation.

Corollary 1. *A center of the group $H = \mathbb{Z}\ltimes_{\varphi}(\mathbb{Z})^n$ is a normal closure of sets: diagonal of Z^n, trivial an element and kernel of action by conjugation that is generated by ρ^{2n} ($\langle\rho^{2n}\rangle \simeq 2n\mathrm{Z}$). In other words,*

$$Z(H) = \langle (1;\ \underbrace{h, h, \ldots, h}_{n}),\ e,\ 2n\mathrm{Z} \ltimes \mathrm{E} \rangle,$$

where $h,\ g \in \mathrm{Z}$. Thus, $Z(H) \simeq 2n\mathbb{Z}\times\mathbb{Z}$. Since the action is defined by conjugation and relation $\rho^{2n}\tau_i\rho^{-2n} = \tau_i$ holds then the element ρ^{2n} commutates with every τ_i. So subgroup stabilise all x_i of Z-space M. Other words subgroup $\langle\rho^{2n}\rangle$ belongs to kernel of action ϕ. Besides the element $(1;\ \underbrace{h, h, \ldots, h}_{n})$ will not be changed by action of conjugation of any element from H because any permutation elements coordinate of diagonal of Z^n does not change it.

We can generalise a result of Meldrum J. [12] because we consider not only the permutation wreath product groups, but the group $\mathcal{A}$ does not have to act upon the set X faithfully, hence $(\mathcal{A}, X) \wr \mathcal{B}$. Recall that an action is said to be faithful if for every $g \in G$, there exists x from G-space X such that $x^g \neq x$. We consider wreath products with no regular actions of the active group $\mathbb{Z}$.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be the $\mathbb{Z}$-space. If an action by conjugation determines a shift of the copies of $\mathbb{Z}$ from the direct product $\mathbb{Z}^n$ then, we have not found a standard wreath product $(\mathbb{Z}, X) \wr \mathbb{Z}$ which is a semidirect product of $\mathbb{Z}$ and $\prod\limits_{x_i \in X} \mathbb{Z}$, i.e. $\mathbb{Z}\ltimes_{\phi}(\mathbb{Z})^n$. Thus, we observe the following corollary holds.

Corollary 2. *The center of the group $\mathbb{Z}\ltimes_{\phi}(\mathbb{Z})^n \simeq (\mathbb{Z}, X) \wr \mathbb{Z}$ consists of normal closure of the diagonal of Z^n, a trivial element and the kernel of action by conjugation, i.e. $n\mathrm{Z}$. In other words,*

$$Z(H) = \langle\, 1;\ \underbrace{h, h, \ldots, h}_{n}),\ e,\ (n\mathbb{Z}, X) \wr \mathbb{E} \rangle \simeq n\mathbb{Z} \times \mathbb{Z},$$

where $h,\ g \in \mathrm{Z}$, $Z(H) \simeq n\mathbb{Z} \times \mathbb{Z}$.

Proof. The proof follows immediately from Corollary 1 by utilising the kernel of action of ϕ. The stabiliser of such an action over the $\mathbb{Z}$-space $X = \{x_1, x_2, \ldots, x_n\}$ is the subgroup $n\mathrm{Z}$. Additionally, the kernel of this action has elements from the diagonal of $\mathbb{Z}^n$.

It should be noted, if we have $G = \mathbb{Z} \underset{X_m}{\wr} \mathbb{Z} \underset{X_n}{\wr} \mathbb{Z}$, where $|X_m| = m$ and $|X_n| = n$, then the action is defined by the shift upon finite set X_n. In this case, we find that $|X| = n$ is not faithful and its kernel is also isomorphic to $n\mathbb{Z}$ since the cyclic shift on the n coordinates is invariant on X. Note that the action is defined by the shift on the finite set X_m is not faithful and its kernel is isomorphic to $m\mathbb{Z}$. Additionally, within this kernel of action is the elements from the diagonal of $\mathbb{Z}^{nm}$ which are isomorphic to $\mathbb{Z}$. Thus, its center is $Z(G) \simeq n\mathbb{Z} \times m\mathbb{Z} \times \mathbb{Z}$ which concludes the proof. □

Remark 2 The center of a group of the form $\mathbb{Z} \ltimes_{\phi} (\mathcal{B})^n \simeq (\mathbb{Z}, X) \wr \mathcal{B}$ generates, by normal closure of: center of diagonal of $\mathcal{B}^n$, trivial an element, and $n\mathrm{Z} \underset{X}{\wr} \mathcal{E}$.

Conclusions

The minimal generating set and the structure of the group $\pi_0 S_{id}(f)$ of the orbit one Morse function have been investigated. The minimal generating set for wreath-cyclic groups have been constructed.

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МІНІМАЛЬНА СИСТЕМА ТВІРНИХ І СТРУКТУРА ВІНЦЕВОГО ДОБУТКУ ГРУП, ФУНДАМЕНТАЛЬНА ГРУПА ОРБІТ ФУНКЦІЇ МОРСА

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РЕЗЮМЕ

Фактор група по комутанту для обмеженого і необмеженого вінцевого добутку знайдено. Досліджено множини твірних для вінцевого добутку.

Структура вінцевого добутку з неточною дією активної групи знайдена. Досліджено мінімальну систему твірних вінцевого добукту і його комутанту, властивості комутанта.

В своїй роботі ми підсилюємо свої попередні результати [18, 21, 20] і будуємо мінімальну систему твірних для комутанту вінцевого добутку як скінченних так і нескінченних груп, також для прямого добутку цих груп. Ми узагальнюємо результ Мелдрума [12] про комутант вінцевого добутку, оскільки розглядаємо не тільки регулярні вінцеві добутки, ми розглядаємо не регулярні вінцеві добутки, де активна група $\mathcal{A}$ діє не точно. Комутант таких груп, система твірних і центр було досліджено нами.

Ми розглядаємо новий клас вінцевоциклічних геометричних груп. Мінімальну систему твірних для цих груп і їх комутант було знайдено.

Ключові слова: *вінцевий добуток груп, комутанта вінцевого добутку, мінімальна система твірних, центр не регулярного вінцевого добутку груп, фундаментальна група орбіт функції Морса, група диффеоморфізмів, що діє на стрічці Мьобіуса.*

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МИНИМАЛЬНАЯ СИСТЕМА ОБРАЗУЮЩИХ И СТРУКТУРА СПЛЕТЕНИЯ ГРУПП, ФУНДАМЕНТАЛЬНАЯ ГРУППА ОРБИТ ФУНКЦИИ МОРСА

РЕЗЮМЕ

Фактор группу по коммутанту для ограниченного и не ограниченного сплетения найдено. Исследованы множества образующих для сплетения.

Структура сплетения с неточным действием активной группы найдена. Исследовано минимальную систему образующих сплетения и его коммутанта, свойства коммутанта.

В этой работе мы усилили предыдущие свои результаты [18, 21, 20] и построили минимальную систему образующих для коммутанат сплетения как конечных так и бесконечных групп, также для прямого произведения этих групп. Мы обобщили результаты Мелдрума [12] о коммутанте сплетения, поскольку рассматриваем не только регулярные сплетения а и не регулярные, где активная группа $\mathcal{A}$ действует не точно. Коммутант таких групп и его система образующи, центр были исследованы нами.

Рассмотрен новый класс венечноцилических геометрических групп. Минимальную систему образующих для этих групп и их коммутант было найдено.

Ключевые слова: *сплетение групп, коммутант сплетения, минимальная система образующих, центр не регулярного сплетения груп, фундаментальная группа орбит функции Морса, группа диффеоморфизмов, ее действие на ленте Мебиуса.*